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CONTENTS

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PROJECTIVITIES OF FINITE PROJECTIVE PLANES.*

By Reinhold Baer.

The group of projectivities of a finite projective plane Π has been investigated fairly thoroughly under the hypothesis that Π is the projective plane over some Galois Field.[1] But once this hypothesis is dropped, the machinery of analytic geometry is not available any more; and comparatively little[2] is known concerning the projectivities of Π.

In the present investigation we undertake to study projectivities of finite projective planes Π without any further hypotheses on the nature of Π. Consequently we shall not be able to make use of such devices as Galois Fields and their automorphisms, the calculus of matrices and so on. In their stead we shall utilize the configuration of the fixed elements of a projectivity and the arithmetical properties of the invariant n of Π—this number n is determined by the fact that every line in Π carries $1 + n$ points and every point is on $1 + n$ lines; it seems to play the same rôle in the theory of finite projective planes as is played by the order in the theory of finite groups.

The system of the fixed elements of a projectivity is closed under the operations of joining and intersecting; and it contains as many points as it contains lines. This last fact is not at all trivial, since its analogue for groups of projectivities fails to be true. The structural properties of the system of fixed elements provide us with a useful principle for classifying projectivities (Section **1**).

Every projectivity effects a permutation in the set of all the points of Π and at the same time in the set of all the points of any given line which it leaves invariant. This makes it possible to apply, in a variety of ways, certain congruences relating orders and characters of permutations. These—together with some applications—we have collected in Section **2**.

With these tools we are able to evolve (in Section **3**) a fairly complete

* Received January 7, 1946; Revised March 25, 1946. Presented to the American Mathematical Society.

[1] See, for instance, Carmichael (1), Chapter XII.

[2] Some such material may be found in Baer (2). The work of Steck (1) is narrower in its scope than ours, since he tacitly presupposes the validity of the Theorem of Desargues, though he neither states nor uses this hypothesis, even in the proofs of some theorems that fail to be true without such a hypothesis.

theory of the projectivities of prime power order. Their simplest and most striking property is the fact that the number of their fixed points is never two. If the order is a power of the prime p, then $p \leqq 1 + n + n^2$ and $1 + n < p$ imply the absence of fixed elements whereas $p = 2$ implies their existence. If the p^i-th power of a projectivity of order a power of p does not leave invariant complete subplanes of Π, then p^{i+1} is a divisor of $(1 + n + n^2)n^2(n^2 - 1)$.

In Section **4** we apply these results to groups Δ of projectivities with the property that the identity is the only projectivity in Δ which leaves invariant a complete subplane of Π. [The best known instance of such a group is the socalled group of special collineations [3] of a projective plane over a Galois Field.] We show that the order of such a group Δ is always a divisor of $(1 + n + n^2)(1 + n)n^3(1 - n)^2$; and if the order of Δ happens to be a divisor of one of the four factors of the above limit, then the nature of the projectivities in Δ may be determined.

Notations.

If Π is a projective plane,[4] then we denote by $P + Q$ the line connecting the two different points P and Q; and we denote by hk the point of intersection of the two different lines h and k.

A projectivity ϕ of Π is a $1 - 1$ correspondence, mapping the point P in Π upon the point $P\phi$ in Π and the line h in Π upon the line $h\phi$ in Π and meeting the following requirement:

$$P \text{ is on } h \text{ if, and only if, } P\phi \text{ is on } h\phi.$$

We are considering finite projective planes. Each of the lines in such a finite projective plane Π carries $n + 1$ points and each point is on $n + 1$ lines. The total number of points (lines) in Π is $1 + n + n^2$. This integer n $(\geqq 2)$ retains its significance throughout.

Projectivities of finite projective planes are necessarily of finite order; and we denote by $o(\phi)$ the *order* of the projectivity ϕ. Since ϕ is completely determined by the permutation it induces in the set of the $1 + n + n^2$ points in Π, it follows that

$$o(\phi) \leqq 1 + n + n^2.$$

[3] For this group, see Carmichael (1), section 93, and Jacobson (1), p. 80.

[4] For the concept of "projective plane" as used here, see Baer (1), (2); Carmichael (1), Chapter XI; Hall (1); Veblen-Young (1).

1. The structure of the system of fixed elements of a projectivity. A point or line left invariant by the projectivity ϕ of Π will be termed a fixed point or a fixed line of ϕ; and we denote by $N(\phi)$ *the total number of fixed points of* ϕ.

THEOREM 1. *$N(\phi)$ is the number of fixed lines of ϕ.*

Proof. The point P and the line h are said to form a *pair* with respect to ϕ, if P is both on h and on $h\phi$; and we denote by $M(\phi)$ the number of distinct pairs with respect to ϕ. Denote furthermore by $N'(\phi)$ the number of fixed lines of ϕ.

If h is a fixed line of ϕ, then P, h is a pair if, and only if, P is on h. Thus $(n+1)N'(\phi)$ is the number of pairs P, h such that h is a fixed line of ϕ. If h is not a fixed line of ϕ, then h and $h\phi$ meet in a well determined point $h(h\phi)$; and $h(h\phi)$, h is the only pair containing h. Hence $1+n+n^2-N'(\phi)$ is the total number of pairs P, h such that h is not a fixed line. Combining these two results we find that

$$M(\phi) = (n+1)N'(\phi) + 1 + n + n^2 - N'(\phi) = nN'(\phi) + 1 + n + n^2.$$

The point P and the line h are said to form a *couple* with respect to ϕ, if h passes both through P and through $P\phi$; and we denote by $M'(\phi)$ the number of distinct couples with respect to ϕ. Since couple is the dual of pair, it follows by dualization that

$$M'(\phi) = nN(\phi) + 1 + n + n^2.$$

The point P and the line h form a pair with respect to ϕ if, and only if, they form a couple with respect to ϕ^{-1}. Hence

$$M(\phi) = M'(\phi^{-1}).$$

But ϕ and ϕ^{-1} have the same fixed elements, proving $N(\phi) = N(\phi^{-1})$. Thus we find finally that

$$\begin{aligned} nN(\phi) + 1 + n + n^2 &= nN(\phi^{-1}) + 1 + n + n^2 = M'(\phi^{-1}) \\ &= M(\phi) = nN'(\phi) + 1 + n + n^2; \end{aligned}$$

and this implies $N(\phi) = N'(\phi)$, as we desired to show.

We denote by $\Phi(\phi)$ *the system of all the fixed elements of the projectivity* ϕ. This system is closed under the operations of connecting points by a line and of intersecting lines. It contains as many points as it contains

Modulo Remark 2

lines [Theorem 1]. Consequently ϕ may belong to one and only one of the following four types.[5]

Type A. *The fixed element free projectivities.*

They are characterized by $N(\phi) = 0$.

Type B. *The generalized elations.*

There exists a fixed line h and a fixed point H *on* h such that every fixed point is on h, every fixed line passes through H. Either H and h are the only fixed elements; or else h is the only fixed line carrying more than one fixed point and H is the only fixed point which is on more than one fixed line. The line h and the point H are, therefore, uniquely determined by the projectivity; and may, consequently, be called its *axis* and *center* respectively.

Type C. *The generalized homologies.*

It will be convenient to distinguish two possibilities.[6]

Type C′. $N(\phi) \neq 3$.

There exists a fixed line h and a fixed point H, not on h, of ϕ with the following properties: every fixed point, not H, is on h; every fixed line, not h, passes through H. The fixed line h carries $N(\phi) - 1 \neq 2$ fixed points; and every fixed line, different from h, (if any), carries exactly two fixed points. The fixed point H is on $N(\phi) - 1 \neq 2$ fixed lines; and every fixed point, different from H, (if any) is on exactly two fixed lines. Thus H and h are uniquely determined by ϕ; and may therefore be termed the *center* and *axis* of ϕ respectively.

Remark 1. It may happen that ϕ is a projectivity of Type B or of Type C′ and that at the same time $N(\phi) = 1$.

Remark 2. Suppose that $N(\phi) = 2$. Then ϕ possesses two fixed points U and V and two fixed lines u and v; and we may select notations in such a way that $u = U + V$ and $U = uv$. Then ϕ is a projectivity of Type B with axis u and center U; and at the same time ϕ is of Type C′ with axis v and center V. If, conversely, the projectivity ϕ is at the same time of Type B and of Type C′, then one verifies readily that $N(\phi) = 2$. This ambiguity

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[5] For an enumeration of the possible types of "degenerate projective planes," see e.g. Hall (1), p. 232. Steck (1) points out a possibility of classifying projectivities according to the number of fixed points.

[6] Sometimes another subdivision of Type C will be found more convenient; see 3, Theorem 3 below.

in notation will not create any confusion, mainly because of **3**, Theorem 4, (2) below.

TYPE C″. $N(\phi)=3$.

Then the system of fixed elements of ϕ is just an ordinary (not degenerate) triangle.[7]

TYPE D. *The projectivities whose system of fixed elements is a projective subplane of* Π.

It is well known [8] that the projectivity ϕ is of Type D if, and only if, there exist four fixed points of ϕ no three of which are collinear. If ϕ is of Type D, then each of its fixed lines carries $i+1$ fixed points; and each of its fixed points is on $i+1$ fixed lines. Furthermore

$$N(\phi)=1+i+i^2 \text{ and } i=\tfrac{1}{2}(-1+(4N(\phi)-3)^{\frac{1}{2}});$$

and we shall say that ϕ is of Type (D,i).

A projectivity ϕ of Type D is completely determined by the system $\Phi(\phi)$ of its fixed elements and by the permutation it induces in the set of points on some fixed line. If ϕ is of Type (D,i), then it has $i+1$ fixed points on every fixed line; and thus it follows that

$$o(\phi)\leqq n-i\leqq n-2, \textit{ if } \phi \textit{ is of Type } (\mathrm{D},i).$$

Remark 3. If the projectivity ϕ belongs to Types C or D, then a proof of Theorem 1 may be effected by a more direct use of the structural properties of the configuration $\Phi(\phi)$. But such a procedure would break down, if ϕ belongs to Types A or B.

Remark 4. One may be tempted to prove an analogue of Theorem 1 for groups of projectivities. If there exist three non-collinear points that are left invariant by all the projectivities in the group Δ, then it is easy to show that Δ has as many fixed points as it has fixed lines (Types C and D). But without this hypothesis the analogue of Theorem 1 fails to be true. For consider as examples the projective plane over a Galois Field and the following groups of projectivities of this plane:

1. All the projectivities which leave invariant a given point P; one fixed point, no fixed line.

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[7] I. e., the three fixed points are not collinear.

[8] Baer (2), Hall (1).

2. All the projectivities which leave invariant two different points P and Q; two fixed points, one fixed line.

3. All the projectivities which leave invariant every point on a given line h; $n+1$ fixed points, one fixed line.

THEOREM 2. *If ϕ^k is of Type* B *or* C′, *then the axis and the center of ϕ^k are fixed elements of ϕ.*

Proof. If x is a fixed element of ϕ^k, then $(x\phi)\phi^k = (x\phi^k)\phi = x\phi$, proving that ϕ maps fixed elements of ϕ^k upon fixed elements of ϕ^k. Thus ϕ induces a projectivity in $\Phi(\phi^k)$. But axis and center of ϕ^k are uniquely determined by projective properties of the configuration $\Phi(\phi^k)$. Hence they are fixed elements of ϕ.

COROLLARY 1. *If ϕ^k is of Type* A *or* B *or* C′, *then ϕ is of Type* A *or* B *or* C *respectively.*

Proof. The system $\Phi(\phi)$ of the fixed elements of ϕ is part of the system $\Phi(\phi^k)$ of the fixed elements of ϕ^k. If ϕ^k is of Type A, then it is fixed element free, implying that ϕ is fixed element free. If ϕ^k is of Type B or C′, then it possesses an axis h and a center H; and it follows from Theorem 2 that h and H are fixed elements of ϕ too. Now one deduces our contention easily from the definitions of the types.

COROLLARY 2. $N(\phi) = N(\phi^k)$ *whenever* $N(\phi^k) < 3$.

Proof. This is obvious if $N(\phi^k) = 0$ (see Corollary 1). It is a consequence of Theorem 2 in case $N(\phi^k) = 1$, since this implies that ϕ^k is of Type B or C′. If, finally, $N(\phi^k) = 2$, then one verifies (as in the proof of Theorem 2) that ϕ maps fixed elements of ϕ^k upon fixed elements of ϕ^k. But one of the two fixed points of ϕ^k is on two fixed lines of ϕ^k whereas the other one is on one and only one. Hence ϕ cannot interchange the two fixed points of ϕ^k, proving that they are fixed points of ϕ. Thus $N(\phi^k) = 2$ implies $N(\phi) = 2$. (See Remark 2.)

COROLLARY 3. *Suppose that* $N(\phi^k) = 3$. *Then*

(a) $N(\phi) \neq 2$.

(b) $N(\phi) = 1$ *implies* $N(\phi^2) = 3$ *and* $k = 2k'$ *for some* k'.

(c) $N(\phi) = 0$ *implies* $N(\phi^3) = 3$ *and* $k = 3k''$ *for some* k''.

(d) *If ϕ^k is of* Type C″ *and if* $o(\phi) \div o(\phi^k)$ *is prime to* 6, *then ϕ is of* Type C″.

Proof. As in the proof of Theorem 2 we show that ϕ induces a permutation of the three fixed points of ϕ^k. The order t of this permutation is 1, 2 or 3 and correspondingly $N(\phi)$ is 3, 1 or 0, since every fixed point of ϕ is a fixed point of ϕ^k. This completes the proofs of (a), (b) and (c), if one remembers that ϕ^k is necessarily a power of ϕ^t and that t is a divisor of $o(\phi)$. To prove (d) we have only to note that $N(\phi) = 3$, if the hypotheses of (d) are satisfied and that therefore ϕ and ϕ^k have the same fixed elements.

COROLLARY 4. *If ϕ^k is of* Type C *and if $o(\phi) \div o(\phi^k)$ is prime to* 3, *then ϕ is of* Type C.

Proof. This is a consequence of Corollary 1 in case ϕ^k is of Type C′. Thus we may assume that ϕ^k is of Type C″ and possesses, therefore, exactly three fixed points P, Q, R which are not collinear. It is a consequence of our hypothesis and Corollary 3, (c) that ϕ possesses fixed points too; and thus, necessarily, one of the three points P, Q, R, say P, is left invariant by ϕ. As to Q and R, they are either both left invariant by ϕ or else they are interchanged by ϕ. In either case the line $R + Q$ is a fixed line of ϕ. Since the fixed point P of ϕ is not on the fixed line $Q + R$ of ϕ, it is shown again that ϕ is of Type C.

2. Characters of permutations. If γ is a permutation of the finite set S, then it is customary[9] to term the number of fixed elements of γ *the character* $\chi(\gamma)$ *of* γ. In this section we shall deduce some formulas on these characters which we shall need for several applications in the future. It may be said here already that the set S may either be the set of all the points of a projective plane or the set of all the points on a fixed line of a projectivity.

THEOREM 1.[10] *If γ is a permutation of a finite set S and p a prime number, then*

$$\chi(\gamma^{p^i}) \equiv \chi(\gamma^{p^j}) \text{ modulo } p^{i+1} \text{ for } 0 \leq i < j.$$

Proof. Every element x in S belongs to a cycle of the permutation γ. This cycle consists of p^i elements in S if, and only if, x is left invariant by γ^{p^i}, but not by $\gamma^{p^{i-1}}$. Since the cycles are mutually exclusive sets, the number of elements in S which are left invariant by γ^{p^i}, but not by $\gamma^{p^{i-1}}$, is a multiple of p^i; and this fact may be restated as follows:

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[9] Speiser (1), p. 18.

[10] This theorem is probably known to many people. But the author did not succeed in locating a convenient reference for it. Speiser (1), Satz 102 is a much deeper theorem; and the deduction of the present formula from it is not quite immediate either.

$$(1) \qquad \chi(\gamma^{p^i}) \equiv \chi(\gamma^{p^{i-1}}) \text{ modulo } p^i.$$

If $i < j$, then we infer from (1) that

$$\chi(\gamma^{p^j}) \equiv \chi(\gamma^{p^{j-1}}) \text{ modulo } p^j$$

and this implies in particular that

$$\chi(\gamma^{p^{j-1}}) \equiv \chi(\gamma^{p^j}) \text{ modulo } p^{i+1}.$$

Consequently we have

$$\chi(\gamma^{p^i}) \equiv \chi(\gamma^{p^{i+1}}) \equiv \cdots \equiv \chi(\gamma^{p^j}) \text{ modulo } p^{i+1},$$

completing the proof.

COROLLARY 1. *If γ is a permutation of the finite set S, p a prime number, $0 \leq i < p$ and $\chi(\gamma^{p^j}) < p^{i+1}$, then*

$$\chi(\gamma^{p^i}) = \chi(\gamma^{p^j}).$$

Proof. Since every fixed element of γ^{p^i} is a fixed element of γ^{p^j}, we may deduce from our hypothesis that

$$0 \leqq \chi(\gamma^{p^j}) - \chi(\gamma^{p^i}) \leqq \chi(\gamma^{p^j}) < p^{i+1}.$$

On the other hand it follows from Theorem 1 that $\chi(\gamma^{p^j}) - \chi(\gamma^{p^i})$ is a multiple of p^{i+1}. Hence $\chi(\gamma^{p^j}) = \chi(\gamma^{p^i})$.

A great number of congruences with composite modulus may be derived from Theorem 1. To obtain a unified derivation of them, we consider the class of number theoretical functions[11] $f(n)$ with the following property:

(*P*) *If k is a positive integer, if p is a prime number, and if $0 \leqq i < j$, then $f(kp^i) \equiv f(kp^j)$ modulo p^{i+1}.*

Noting that $\gamma^{kp^h} = (\gamma^k)^{p^h}$ we infer from Theorem 1 that the characters $\chi(\gamma^i) = t(i)$ of the powers of a given permutation are number-theoretical functions with Property (*P*). Another example is furnished—by Euler's Theorem—by the powers of a given integer. Finally it may be worth noting that the sum, difference and product of number theoretical functions with the property (*P*) are again number theoretical functions with the property (*P*) so that *these functions form a ring.*

In order to simplify the enunciation of our next theorem, we have to introduce a few symbols. If m is a positive integer, then we denote by m^* the product of all the different prime divisors of m. Thus m^* *is the* l. c. m.

[11] These are integral valued functions of an integral variable.

of all the squarefree divisors of m. Next we need a generalization of Möbius' function, namely the function $\nu_D(d)$ which we define for divisors d of the integer D only by the following rules:

$\nu_D(d) = 0$, if d and Dd^{-1} are not relatively prime;

$\nu_D(d) = 1$, if d and Dd^{-1} are relatively prime and if the number of different prime divisors of d is even;

$\nu_D(d) = -1$, if d and Dd^{-1} are relatively prime and if the number of different prime divisors of d is odd.

THEOREM 2. *If D is a divisor of the positive integer m and a multiple of m^*, and if the number theoretical function $f(n)$ meets the requirement (P), then*

$$\sum_{d/D} \nu_D(d) f(md^{-1}) \equiv 0 \text{ modulo } mD^{-1}m^*.$$

Proof. Suppose that the prime number p is a divisor of m. Denote by p^h the highest power of p, dividing D; and denote by p^{h+k} the highest power of p, dividing m. Then $0 < h$, since p is a divisor of m^* and, therefore, of the multiple D of m^*; and $0 \leq k$, since D is a divisor of m.

(i) If the divisor d of D is prime to p, then $\nu_D(d) = -\nu_D(dp^h)$.

Since d and p are relatively prime, and since p^h is the highest power of p, dividing D, it follows that the g. c. d. of d and Dd^{-1} is the same as the g. c. d. of dp^h and $D(dp^h)^{-1}$. Hence $\nu_D(d) = 0$ if, and only if, $\nu_D(dp^h) = 0$. Since d is prime to p, and since h is positive, dp^h is divisible by just one prime number more than d; and now (i) is an immediate consequence of the definition of ν_D.

(ii) If the divisor d of D is prime to p, then

$$\nu_D(d) f(md^{-1}) + \nu_D(dp^h) f(m(dp^h)^{-1}) \equiv 0 \text{ modulo } p^{k+1}.$$

Since d and p are prime, dp^{k+h} is a divisor of m; and hence $j = md^{-1}p^{-k-h}$ is an integer. Now we deduce from (i) and Property (P) that

$$\begin{aligned} \nu_D(d) f(md^{-1}) + \nu_D(dp^h) f(m(dp^h)^{-1}) &\equiv \nu_D(d)[f(jp^{k+h}) - f(jp^k)] \\ &\equiv 0 \text{ modulo } p^{k+1}. \end{aligned}$$

(iii)
$$\sum_{d/D} \nu_D(d) f(md^{-1}) \equiv 0 \text{ modulo } p^{k+1}.$$

Every divisor of D has the form dp^j where d is prime to p and $0 \leq j \leq h$. The divisors of D that are prime to p are exactly the divisors of $D' = Dp^{-h}$. Noting finally that $\nu_D(dp^j) = 0$ whenever d is a divisor of D' and $0 < j < h$, since in this case both dp^j and $D(dp^j)^{-1}$ are divisible by p, we find that

$$\sum_{d/D} \nu_D(d)f(md^{-1}) = \sum_{d/D'} [\nu_D(d)f(md^{-1}) + \nu_D(dp^h)f(m(dp^h)^{-1})];$$

and (iii) is an immediate consequence of (ii).

But p^{k+1} is the highest power of the prime number p which divides $mD^{-1}m^*$. Thus it follows from (iii) that every prime power divisor of $mD^{-1}m^*$ is a divisor of $\sum_{d/D} \nu_D(d)f(md^{-1})$. Hence $mD^{-1}m^*$ itself is a divisor, completing the proof of this theorem.

Remark 1. Note that $d = 1$ is always a divisor of D and $\nu_D(1) = 1$. Thus $f(m)$ itself appears always as a term in the sum of Theorem 2.

Remark 2. We may let, in particular, $D = m$ in Theorem 2. Then the sum ranges over all the divisors of m and the modulus of the congruence is m^*.

The most interesting special case of Theorem 2 involves in its statement the use of Möbius' function $\mu(d)$. It should be remembered that $\mu(d) = 0$ if d is not squarefree, that $\mu(d) = 1$, if d is the product of an even number of different primes and that $\mu(d) = -1$, if d is the product of an odd number of different primes.

COROLLARY 2. *If the number-theoretical function $f(m)$ meets the requirement (P), then*

$$\sum_{d/m} \mu(d)f(md^{-1}) \equiv 0 \text{ modulo } m.$$

Proof. If d is a divisor of m, then $\mu(d) \neq 0$ is a necessary and sufficient condition for d to be a divisor of m^*. If d is a divisor of m^*, then d and m^*d^{-1} are relatively prime so that $\nu_{m^*}(d) = \mu(d) \neq 0$. From these remarks we infer that

$$\sum_{d/m} \mu(d)f(md^{-1}) = \sum_{d/m^*} \nu_{m^*}(d)f(md^{-1}).$$

Considering now the special case $D = m^*$ of Theorem 2, we find that this sum is divisible by $mD^{-1}m^* = m$.

COROLLARY 3. *If γ is a permutation of the finite set S, if $o(\gamma) = m$ is the order of the permutation γ and M the number of elements in S, and if $\chi(\gamma^{mp^{-1}}) < q$ for every prime divisor p of m and every prime divisor q of mp^{-1}, then $\chi(\gamma) \equiv M$* modulo m.

Proof. If d is a divisor of m, $1 < d$, and if q is a prime divisor of md^{-1}, then it follows from our hypothesis that $\chi(\gamma^{md^{-1}}) < q$. Hence we infer from Corollary 1 that $\chi(\gamma^{md^{-1}}) = \chi(\gamma^{m(dq)^{-1}})$. Now it follows by complete induction that $\chi(\gamma) = \chi(\gamma^{mi^{-1}})$ whenever $i \neq 1$ is a divisor of m. Remembering that

$\chi(\gamma^m) = \chi(1) = M$ and that $\sum_{d/m} \mu(d) = 0$ for $m \neq 1$, we deduce now from Corollary 2 (and Theorem 1) that $\chi(\gamma) \equiv M$ modulo m.

If ϕ is a projectivity of the projective plane Π, then ϕ effects a permutation of the $1 + n + n^2$ points in Π; and the number $N(\phi)$ of the fixed points of ϕ is the character of this permutation. It is now quite clear how to use the results of the present section for obtaining fixed point formulas for projectivities. The following application, however, does not seem to be without interest.

If the projectivity ϕ is of Type D, then there exists a uniquely determined integer $i(\phi)$ such that every fixed line of ϕ carries $i(\phi) + 1$ fixed points of ϕ. Furthermore every power of ϕ is of Type D too so that the numbers $i(\phi^k)$ are well determined.

THEOREM 3. *If the projectivity ϕ is of* Type D, *then*

$$\sum_{d/o(\phi)} \mu(d) i(\phi^{o(\phi)d^{-1}}) \equiv 0 \text{ modulo } o(\phi).$$

Proof. Consider a fixed line h of ϕ. Then ϕ effects a permutation of the $n + 1$ points of h; and it is a consequence of Theorem 1 that the number-theoretical function $i(\phi^k) + 1$ meets the requirement (P). Hence $i(\phi^k)$ meets the requirement (P) too; and now our Theorem is a consequence of Corollary 2.

3. Projectivities of order a power of a prime.

Apart from the number $N(\phi)$ of the fixed points (lines) of the projectivity ϕ we shall make use of *the number $N(\phi, h)$ of all the fixed points of ϕ which are situated on the line h.* The discussion of this section will be based on the following proposition.

THEOREM 1. *If the projectivity ϕ of the finite projective plane Π is of order $o(\phi) = p^m$ for p a prime, and if $0 \leqq i < m$, then*

(a) $N(\phi^{p^i}) \equiv 1 + n + n^2$ modulo p^{i+1}; *and*

(b) $N(\phi^{p^i}, h) \equiv 1 + n$ modulo p^{i+1} *for every fixed line h of ϕ.*

Proof. ϕ^{p^i} effects a permutation of the $1 + n + n^2$ points of Π whose character is $N(\phi^{p^i})$. Hence it follows from **2**, Theorem 1, that $N(\phi^{p^i}) \equiv N(\phi^{p^m}) \equiv 1 + n + n^2$ modulo p^{i+1}, proving (a). If h is a fixed line of ϕ, then ϕ and its powers induce permutations of the $n + 1$ points on h; and the characters of these permutations are the numbers $N(\phi^j, h)$. Hence it follows from **2**, Theorem 1, that $N(\phi^{p^i}, h) \equiv N(\phi^{p^m}, h) \equiv n + 1$ modulo p^{i+1}, proving (b).

Corollary 1. *If the projectivity ϕ is of prime power order $o(\phi) = p^m$, and if $n + 1 < o(\phi)$, then*

(a) $N(\phi) = 0$ *or* 1; *and*

(b) $N(\phi) = 1$ *if, and only if,* $\phi^{p^{m-1}}$ *is a perspectivity; and*

(c) $m = 1$ *implies* $N(\phi) = 0$.

Proof. If h is a fixed line of ϕ, then ϕ effects a permutation of the $n + 1$ points on h. Thus it follows from **2**, Corollary 1, (or Theorem 1, (b)) that $N(\phi^{p^{m-1}}, h) = n + 1$. Hence every point on the fixed line h of ϕ is a fixed point of $\phi^{p^{m-1}}$. But it is impossible that a projectivity, not 1, leaves invariant all the points on two different lines. Hence $N(\phi) \leq 1$, proving (a). If $N(\phi) = 1$, then the points on the fixed line of ϕ are fixed points of $\phi^{p^{m-1}}$; and it is well known[12] that a projectivity, not 1, which leaves invariant all the points on a certain line is a perspectivity. Thus $\phi^{p^{m-1}}$ is a perspectivity whenever $N(\phi) = 1$. If, conversely, $\phi^{p^{m-1}}$ is a perspectivity, then it follows from **1**, Theorem 2, that its axis and center are fixed elements of ϕ so that $N(\phi) \neq 0$; and this implies $N(\phi) = 1$ by (a), completing the proof of (b). Since perspectivities leave invariant more than 1 point, $m = 1$ and $N(\phi) = 1$ are incompatible by (b); and (c) is a consequence of (a).

Theorem 2A. *If $o(\phi) = p^m$ for p a prime, and if ϕ^{p^i} is fixed element free, then* $1 + n + n^2 \equiv 0$ modulo p^{i+1}.

This is a consequence of the fact that by Theorem 1, (a)

$$0 \equiv N(\phi^{p^i}) \equiv 1 + n + n^2 \text{ modulo } p^{i+1}.$$

Corollary 2. *Projectivities of order a power of 2 possess fixed elements.*

For in this case we may infer from Theorem 1, (a) that $N(\phi)$ is odd, since $1 + n + n^2$ is always odd.

Corollary 3 *If $1 + n + n^2$ is a prime, then $o(\phi) = 1 + n + n^2$ is a necessary and sufficient condition for ϕ to be a fixed element free projectivity of prime power order.*

Proof. If $N(\phi) = 0$ and $o(\phi) = p^m$ for p a prime, then it follows from Theorem 2A that p is a prime divisor of the prime number $1 + n + n^2$ so that $p = 1 + n + n^2$. Hence ϕ is a permutation of p points whose order is $p^m \neq 1$, proving that $m = 1$ and $o(\phi) = 1 + n + n^2$. If, conversely, ϕ is

[12] E. g. Baer (1), p. 140, Corollary 2. 3.

of order the prime number $p = 1 + n + n^2$, then it effects a cyclic permutation of the p points in Π, implying $N(\phi) = 0$.

Remark 1. If Π happens to be the projective plane over a Galois Field $GF(n)$, then there exists [13] a projectivity γ which effects a cyclic permutation of the $1 + n + n^2$ points in Π. Clearly γ and all its powers are fixed element free, showing the impossibility of improving Theorem 2A. The order of γ is the greatest possible order of projectivities of Π, since every projectivity effects a permutation of the $1 + n + n^2$ points whose order certainly cannot exceed $1 + n + n^2$.

THEOREM 2B. *If $o(\phi) = p^m$ for p a prime and ϕ^{p^t} is of* Type B, *then* $n^2 \equiv 0$ modulo p^{t+1}; *and* $n \equiv 0$ modulo p^{t+1}, *if either $N(\phi) \neq 1$ or $N(\phi^{p^t}) = 1$.*

Proof. Since ϕ^{p^t} is of Type B, all its fixed points are on its axis h and all its fixed lines pass through its center H. It follows from **1**, Theorem 2, that H and h are fixed elements of ϕ too. Now we deduce from Theorem 1 that

$$1 + n + n^2 \equiv N(\phi^{p^t}) \equiv N(\phi^{p^t}, h) \equiv n + 1 \text{ modulo } p^{t+1}.$$

Consequently $n^2 \equiv 0$ modulo p^{t+1}. If furthermore $N(\phi^{p^t}) = 1$, then $n \equiv 0$ modulo p^{t+1} may be deduced from the above congruences.

Assume, finally, that $N(\phi) \neq 1$. Since h is a fixed line of ϕ, there exists a fixed line w of ϕ such that $w \neq h$. But every fixed element of ϕ is a fixed element of ϕ^{p^t}; and H is consequently the one and only fixed point of ϕ^{p^t} which is on w. Thus it follows from Theorem 1, (b) that

$$1 \equiv N(\phi^{p^t}, w) \equiv n + 1 \text{ modulo } p^{t+1}$$

or $n \equiv 0$ modulo p^{t+1}, completing the proof.

Remark 2. It is easy to construct projectivities of prime power order $p^m = n^2$ whose p^{m-1}-st power is of Type B (in which case p^m is not a divisor of n).

COROLLARY 4. *Suppose that ϕ is a projectivity of prime power order and that n is a prime number. Then ϕ is of* Type B *if, and only if, $o(\phi) = n$ or n^2.*

Proof. If $o(\phi) = p^m$ for p a prime number, and if ϕ is of Type B, then it follows from Theorem 2B that p is a divisor of n. But n is a prime

[13] Singer (1), p. 379.

number, proving $n = p$. If h is a fixed line of ϕ and $0 \leq i < m$, then it follows from Theorem 1, (b) that

$$1 \equiv 1 + n \equiv N(\phi^{p^i}, h) \text{ modulo } p.$$

But the only numbers between 0 and $n + 1$ that are congruent to 1 modulo $p = n$ are 1 and $n + 1$, proving that $N(\phi^{p^i}, h)$ is either 1 or $n + 1$. Since h is a fixed line of ϕ and, therefore, of ϕ^{p^i}, this implies that ϕ^{p^i} is of Type B too. Hence it follows from Theorem 2B that p^m is a divisor of $n^2 = p^2$ so that $o(\phi)$ is either n or n^2.

Assume, conversely, that $o(\phi) = n$. Then it follows from Theorem 1, (a) that $N(\phi) \equiv 1 + n + n^2 \equiv 1$ modulo n so that, in particular, $N(\phi) \neq 0$. If h is a fixed line of ϕ, then it follows from Theorem 1, (b) that $N(\phi, h) \equiv n + 1 \equiv 1$ modulo n; and it follows as before that $N(\phi, h)$ is either 1 or $n + 1$. But a projectivity possessing fixed elements whose fixed lines carry 1 or $n + 1$ fixed points, is a projectivity of Type B.

If, finally, $o(\phi) = n^2$, then $o(\phi^n) = n$; and it follows from the result of the preceding paragraph of the present proof that ϕ^n is of Type B. Hence it is a consequence of **1**, Corollary 1, that ϕ is of Type B.

If we had at the same time $o(\phi) = n^2$ and $N(\phi) \neq 1$, then there would exist a fixed line k of ϕ, not the axis of ϕ (nor of ϕ^n). Thus the center of ϕ and ϕ^n would be the only fixed point on k; and we could infer from Theorem 1, (b) that

$$1 \equiv N(\phi^n, k) \equiv n + 1 \text{ modulo } n^2$$

which is impossible. Thus we have not only completed the proof of Corollary 4, but also of the following proposition.

COROLLARY 4′. *If n is a prime number, and if $o(\phi) = n$, then ϕ is either an elation (in the strict sense) or else $N(\phi) = 1$; and $o(\phi) = n^2$ implies $N(\phi) = 1$.*

THEOREM 2C′. *Suppose that ϕ is of prime power order $o(\phi) = p^m$, and that ϕ^{p^i} is of* Type C′.

(a) *If $N(\phi) \neq 1$, then $n \equiv 1$ modulo p^{i+1}.*

(b) *If p is odd and $N(\phi) = 1$, then $n \equiv -1$ modulo p^{i+1} and $N(\phi^{p^i}) = 1$.*

(c) *If $p = 2$ and $N(\phi^{2^i}) = 1$, then $n \equiv -1$ modulo 2^{i+1}.*

(d) *If $p = 2$ and $N(\phi) = 1 \neq N(\phi^{2^i})$, then either $N(\phi^2) \neq 1$ and $n \equiv 1$ modulo 2^i or else $N(\phi^{2^{i-1}}) = 1$ and $n \equiv -1$ modulo 2^i.*

Proof. Since ϕ^{p^i} is of Type C′, it possesses an axis h and a center H

not on h, which are by **1**, Theorem 2, fixed elements of ϕ. All the fixed points of ϕ^{p^i} with the exception of H are on h; and all the fixed lines of ϕ^{p^i}, not h, pass through H. Hence it follows from Theorem 1 that

$$1 + n + n^2 \equiv N(\phi^{p^i}) \equiv N(\phi^{p^i}, h) + 1 \equiv (n + 1) + 1 \text{ modulo } p^{i+1} \text{ or}$$

(e) $\quad n^2 \equiv 1 \text{ modulo } p^{i+1}$.

If $N(\phi) \neq 1$, then there exists a fixed line w of ϕ which is different from the axis h of ϕ^{p^i}, since h is a fixed line of ϕ. This line w carries two and only two fixed points of ϕ^{p^i}, namely H and the intersection wh. Hence it follows from Theorem 1, (b) that

$$2 \equiv N(\phi^{p^i}, w) \equiv n + 1 \text{ modulo } p^{i+1} \quad \text{or} \quad n \equiv 1 \text{ modulo } p^{i+1},$$

proving (a).

Suppose finally that $N(\phi) = 1$. Then we infer from (e) and Theorem 1, (a) that

$$1 \equiv N(\phi) \equiv 1 + n + n^2 \equiv 2 + n \text{ modulo } p \quad \text{or} \quad n \equiv -1 \text{ modulo } p.$$

If p is odd, then this implies that p and $n - 1$ are relatively prime. From (e) we infer that $(n + 1)(n - 1) \equiv n^2 - 1 \equiv 0 \text{ modulo } p^{i+1}$; and this implies $n \equiv -1 \text{ modulo } p^{i+1}$, proving the first part of (b). If $N(\phi^{p^i})$ were different from 1, then we could apply (a) on $\phi^{p^i} = (\phi^{p^i})^{p^0}$; and we would obtain $n \equiv 1 \text{ modulo } p$ which is incompatible with $n \equiv -1 \text{ modulo } p^{i+1}$, since $p \neq 2$. Hence $N(\phi^{p^i}) = 1$, completing the proof of (b).

If $p = 2$ and $N(\phi^{2^i}) = 1$, then the fixed line h of ϕ does not carry any fixed points of ϕ^{2^i}; and we infer from Corollary 1, (b) that

$$0 \equiv N(\phi^{2^i}, h) \equiv n + 1 \text{ modulo } 2^{i+1},$$

proving (c).

If finally $p = 2$ and $N(\phi^{2^i}) \neq 1 = N(\phi)$, then there exists a positive integer $k \leq i$ such that $N(\phi^{2^k}) \neq 1 = N(\phi^{2^{k-1}})$. Applying (a) on ϕ^{2^k} and $(\phi^{2^k})^{2^{i-k}} = \phi^{2^i}$, we find that $n \equiv 1 \text{ modulo } 2^{i-k+1}$; and applying (c) on ϕ and $\phi^{2^{k-1}}$ we find that $n \equiv -1 \text{ modulo } 2^k$. Since it is impossible that n is modulo 4 congruent to $+1$ as well as to -1, it follows that $k = 1$ or $i = k$. If $k = 1$, then $N(\phi^2) \neq 1$; and applying (a) upon ϕ^2 and $(\phi^2)^{2^{i-1}} = \phi^{2^i}$ we see that $n \equiv 1 \text{ modulo } 2^i$. If $i = k$, then $N(\phi^{2^{i-1}}) = 1$; and it follows from (c) that $n \equiv -1 \text{ modulo } 2^i$, completing the proof.

THEOREM 2C″. *Suppose that ϕ is of prime power order $o(\phi) = p^m$, and that ϕ^{p^i} is of* Type C″.

(a) *If* $p \neq 2, 3$, *then* $n \equiv 1$ modulo p^{i+1} *and* $N(\phi) = 3$.

(b) *If* $p = 2$, *then* $n \equiv 1$ modulo 2^{i+1} *and* $N(\phi) = 1$ *or* 3.

(c) *If* $p = 3$, *then* $N(\phi) \neq 0$ *implies* $n \equiv 1$ modulo 3^{i+1} *and* $N(\phi) = 3$; *and* $N(\phi) = 0$ *implies* $n \equiv 1$ modulo 3^i *and* $N(\phi^3) = 3$.

Proof. The system $\Phi(\phi^{p^i})$ is an ordinary, non degenerate triangle, since ϕ^{p^i} is of Type C″; and the system of fixed elements of ϕ^{p^j} for $0 \leq j \leq i$ is part of this triangle. We show first:

(d) $N(\phi) \neq 0$ *implies* $n \equiv 1$ modulo p^{i+1}.

For if w is a fixed line of ϕ, then w carries two, and only two, fixed points of ϕ^{p^i}. Hence it follows from Theorem 1, (b) that $2 \equiv N(\phi^{p^i}, w) \equiv n + 1$ modulo p^{i+1}, proving (d).

If $p \neq 2, 3$, then we infer from **1**, Corollary 3 that $N(\phi) = 3$; and (a) is a consequence of (d). If $p = 2$, then it follows from **1**, Corollary 3 that $N(\phi) = 1$ or 3; and (b) is a consequence of (d). If finally $p = 3$, then we infer from **1**, Corollary 3 that $N(\phi) = 0$ or 3 and that $N(\phi) = 0$ implies $N(\phi^3) = 3$. Now (c) is obtained by applying (d) on ϕ^3, if $N(\phi) = 0$, and by applying (d) on ϕ, if $N(\phi) \neq 0$.

COROLLARY 5. *If* $n - 1$ *is a prime number, then the projectivity* ϕ *of odd prime power order* $o(\phi) = p^m$ *has the following properties.*

(a) *If* ϕ *is of* Type C *and* $N(\phi) \neq 1$, *then* $o(\phi) = n - 1$.

(b) *If* $o(\phi) = n - 1$ *and* $n \neq 4$, *then* ϕ *is of* Type C *and* $N(\phi) \neq 1$.

Proof. Suppose first that ϕ is of Type C and that $N(\phi) \neq 1$. Then it follows from Theorem 2C′, (a) and Theorem 2C″ that $n \equiv 1$ modulo p. Thus p is a divisor of the prime number $n - 1$, proving $p = n - 1$. Consider a fixed line h of ϕ. Then it follows from Theorem 1, (b) that

$$N(\phi^{p^i}, h) \equiv n + 1 \equiv (n - 1) + 2 \equiv p + 2 \equiv 2 \text{ modulo } p.$$

Thus $N(\phi^{p^i}, h)$ is a number between 0 and $n + 1$ which is congruent to 2 modulo $p = n - 1$. Hence $N(\phi^{p^i}, h)$ is either 2 or $n + 1$, for every fixed line h *of* ϕ. Noting $1 < N(\phi) \leqq N(\phi^{p^i})$ and noting the fact that the fixed points of ϕ, and therefore those of ϕ^{p^i}, are not all collinear, it follows that $\phi^{p^{m-1}}$ is of Type C. Hence it follows from $N(\phi) \neq 1$ and Theorem 2C′, (a) and Theorem 2C″ that

$$n \equiv 1 \text{ modulo } p^m$$

so that $n - 1 = p$ is divisible by p^m. Hence $m = 1$, proving that $o(\phi) = p = n - 1$.

Assume, conversely, that $o(\phi) = n - 1 \neq 3$. Since $n - 1 = p$ is a prime number, it follows from Theorem 1, (a) that

$$N(\phi) \equiv 1 + n + n^2 \equiv 3 \text{ modulo } p.$$

If $N(\phi)$ were 0, then this would imply $n - 1 = p = 3$; and we excluded this possibility in our hypothesis. Hence $N(\phi) \neq 0$ and there exist fixed lines of ϕ. If h is a fixed line of ϕ, then it follows from Theorem 1, (b) that

$$N(\phi, h) \equiv n + 1 \equiv 2 \text{ modulo } p.$$

But $N(\phi, h)$ is a number between 0 and $n + 1 = p + 2$, proving that $N(\phi, h)$ is either 2 or $n + 1$. Consequently $N(\phi) \neq 1$ and ϕ is of Type C, completing the proof.

Remark 3. If Π is the projective plane over the Galois Field GF(4), then $n = 4$. If r is a number neither 0 nor 1 in GF(4), then the projectivity ϕ mapping the point with coordinates (x_0, x_1, x_2) upon the point with coordinates (rx_2, x_0, x_1) is of order $3 = n - 1$ and is fixed point free, showing that the hypothesis $n \neq 4$ cannot be omitted in (b). Note that it follows from the proof that we could have substituted the hypothesis $N(\phi) \neq 0$ for the hypothesis $n \neq 4$ in part (b).

COROLLARY 6. *If $n + 1$ is a prime number, then the following properties of the projectivity ϕ of prime power order $o(\phi) = p^m$ imply each other.*

(a) $o(\phi) = n + 1$.

(b) ϕ *is of* Type C *and* $N(\phi) = 1$.

Proof. Assume first that ϕ is of Type C and that $N(\phi) = 1$. Then it follows from Theorem 2C′, (b) and (c) that $n \equiv -1$ modulo p. Thus p is a divisor of the prime number $n + 1$ and hence $p = n + 1$. This implies in particular $p \neq 2$, since $2 \leqq n$. If h is the fixed line of ϕ, then it follows from Theorem 1, (b) that

$$N(\phi^{p^i}, h) \equiv n + 1 \equiv p \equiv 0 \text{ modulo } p$$

so that $N(\phi^{p^i}, h)$ is either 0 or $n + 1$. Since $N(\phi, h) = 0$, and since ϕ possesses a fixed point not on h, it would follow from $N(\phi^{p^i}, h) = n + 1$ that $i \neq 0$ and that ϕ^{p^i} is a homology (Type C′). If $N(\phi^{p^i}, h) = 0$, then it follows from the definitions of the types that ϕ^{p^i} is of Type C′ and $N(\phi^{p^i}) = 1$. In either case we deduce from Theorem 2C′, (b) that $n \equiv -1$ modulo p^m so that p^m is a divisor of $n + 1 = p$. Hence $m = 1$ and $o(\phi) = p = n + 1$, proving that (a) is a consequence of (b).

Assume, conversely, that $o(\phi) = n + 1 = p$. Then p is odd; and we infer from Theorem 1, (a) that

$$N(\phi) \equiv 1 + n + n^2 \equiv 1 \text{ modulo } p$$

so that in particular $N(\phi) \neq 0$. If h is a fixed line of ϕ, then it follows from Theorem 1, (b) that

$$N(\phi, h) \equiv n + 1 \equiv 0 \text{ modulo } p$$

so that $N(\phi, h)$ is either 0 or $n + 1$. But if $N(\phi, h)$ were $n + 1$, then ϕ would be a perspectivity and there would exist fixed lines of ϕ carrying 1 or 2 fixed points, contradicting the fact, just proven, that every fixed line of ϕ carries 0 or $n + 1$ fixed points. Thus $N(\phi, h) = 0$ for every fixed line of ϕ, proving that $N(\phi) = 1$ and that ϕ is of Type C. Hence (b) is a consequence of (a).

Remark 4. Every power of a prime may be the integer n of a projective plane; but it is not known at present whether any other integer may be the n of a projective plane. If n is a prime power and $n - 1$ a prime, then either $n = 3$ or $n = 2^j$; and if n is a prime power and $n + 1$ a prime, then $n = 2^r$. We note that $n - 1$ is a prime, if $n = 3, 4, 8, 32, 128$ and that $n + 1$ is a prime for $n = 4, 16, 256$.

Remark 5. Denote by Z the group of projectivities of the projective plane Π which is generated by the perspectivities (in the strict sense of the word). This group has sometimes been termed [14] *the group of special collineations.* If Π is in particular the projective plane over the Galois Field GF(n) consisting of exactly n elements, then it is well known [15] that the order of Z is $(1 + n + n^2)(1 + n)n^3(1 - n)^2$ and that the only projectivity of Type D in Z is the identity. If the prime number p is a divisor of the order Z, then Z contains an element of the order p. Thus it follows from the Corollaries 5 and 6 that the results obtained in Theorems 2C′ and 2C″ are "best" results. The method employed by Singer (1) may be used to prove the existence of a projectivity of order $n + 1$ in Z.

In the following proposition we give a summary of some of the preceding results which seems of particular interest in the light of the preceding Remark 5.

COROLLARY 7. *If ϕ is of prime power order $o(\phi) = p^m$, and if ϕ^{p^i} is not of* Type D, *then,*

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[14] Jacobson (1), p. 80.

[15] Carmichael (1), p. 358.

(a) p^{i+1} *is a divisor of* $n^2(1+n+n^2)(n-1)(n+1)$; *and*
(b) $N(\phi) \equiv 0$ *or* 1 *or* 3 modulo p^{i+1}.

Proof. If ϕ^{p^i} is of Types A or B, then we infer from Theorems 2A and 2B that p^{i+1} is a divisor of $1+n+n^2$ or n^2 respectively. If ϕ^{p^i} is of Type C′, then we deduce from the statement (e), derived in the course of the proof of Theorem 2C′ that p^{i+1} is a divisor of n^2-1. If finally ϕ^{p^i} is of Type C″, then it follows from Theorem 2C″, (a) and (b) that p^{i+1} is a divisor of $n-1$, unless $p=3$ and $N(\phi)=0$. In the latter case 3^i is a divisor of $n-1$ and 3 is a divisor of $1+n+n^2$ so that 3^{i+1} is a divisor of $(n-1)(1+n+n^2)$, completing the proof of (a). (b) is readily deduced from the Theorems 2, since the exceptional cases of these theorems always involve that $N(\phi)=0$ or 1 or 3.

The extent to which the geometrical properties of a projectivity of order a power of p are determined by the arithmetical properties of the integers n and p will be made strikingly clear by the following theorem. Important applications will be made in the next section.

THEOREM 3. *Suppose that* ϕ *is of prime power order* $o(\phi)=p^m$, *and that* ϕ^{p^i}, *for* $i<m$, *is not of* Type D.

(A) *If* $p\neq 3$, *then the following properties imply each other.*
(A, 1) ϕ^{p^i} *is of* Type A;
(A, 2) $1+n+n^2\equiv 0$ modulo p^{i+1};
(A, 3) $1+n+n^2\equiv 0$ modulo p;
(A, 4) ϕ *is of* Type A.
(B) *The following properties imply each other.*
(B, 1) ϕ^{p^i} *is of* Type B;
(B, 2) $n^2\equiv 0$ modulo p^{i+1};
(B, 3) $n\equiv 0$ modulo p;
(B, 4) ϕ *is of* Type B.
(C*) *If* p *is odd, then the following properties imply each other.*
(C*, 1) ϕ^{p^i} *is of* Type C *and* $N(\phi^{p^i})=1$;
(C*, 2) $n\equiv -1$ modulo p^{i+1};
(C*, 3) $n\equiv -1$ modulo p;
(C*, 4) ϕ *is of* Type C *and* $N(\phi)=1$.
(C) *If* $p\neq 2, 3$, *then the following properties imply each other.*
(C, 1) ϕ^{p^i} *is of* Type C *and* $N(\phi^{p^i})\neq 1$;
(C, 2) $n\equiv 1$ modulo p^{i+1};
(C, 3) $n\equiv 1$ modulo p;
(C, 4) ϕ *is of* Type C *and* $N(\phi)\neq 1$.

Proof. It is a consequence of Theorem 2A that (A, 1) implies (A, 2); and it is obvious that (A, 2) implies (A, 3). It is a consequence of Theorem 2B that (B, 1) implies (B, 2); and it is obvious that (B, 2) implies (B, 3), since p is a prime number. It is a consequence of Theorem 2C′, (b) and the oddness of p that (C*, 1) implies (C*, 2); and it is obvious that (C*, 2) implies (C*, 3). If $p \neq 2, 3$, then it follows from Theorem 2C′, (a) and from Theorem 2C″, (a) that (C, 1) implies (C, 2); and it is obvious that (C, 2) implies (C, 3). If, finally, $p = 2$ or 3, then we have $n^2 \equiv 1$ modulo p in the last two cases.

We shall make use of these implications during the remainder of the present proof.

If $p \neq 3$ and if (A, 3) is valid, then (B, 3), (C*, 3) and (C, 3) cannot hold, since they would imply that $1 + n + n^2$ is congruent to 1 or 3 modulo p. Thus it follows from the implications already verified that ϕ^{p^i} cannot be of Types B or C. Hence ϕ^{p^i} is of Type A, proving that (A, 1) is a consequence of (A, 3) and $p \neq 3$. But (A, 1) obviously implies (A, 4); and that (A, 4) implies (A, 3), is one of the implications proved in the first paragraph of this proof, completing the proof of (A).

If (B, 3) is satisfied by n, then (A, 3), (C*, 3) and (C,3) are not satisfied. Thus ϕ^{p^i} cannot be of Types A or C. Hence it is of Type B, proving that (B, 1) is a consequence of (B, 3). By the same argument we see that ϕ is of Type B, proving that (B, 4) is a consequence of (B, 3). But (B, 4) implies (B, 3) by the results of the first paragraph of this proof; and (B, 3) implies (B, 1), completing the proof of (B).

If p is odd, and if (C*, 3) is satisfied by n, then neither (A, 3) nor (B, 3) nor (C, 3) are satisfied by n. Thus ϕ and ϕ^{p^i} cannot be of Types A or B; and if they are of Type C, then $N(\phi) = N(\phi^{p^i}) = 1$. Hence (C*, 4) and (C*, 1) are consequences of (*C**, 3); and it is clear how to complete the proof of (C*).

If, finally, $p \neq 2, 3$, and if (C, 3) is satisfied by n, then $1 + n + n^2 \equiv 3$ modulo p so that neither (A, 3) nor (B, 3) nor (C*, 3) is satisfied by n. Thus ϕ and ϕ^{p^i} cannot be of Types A or B; and if they are of Type C, then $N(\phi)$ and $N(\phi^{p^i})$ are both different from 1. Consequently (C, 3) implies (C, 1) and (C, 4); and it is clear how to complete the proof of (C).

Remark 6. If Π is the projective plane over the Galois Field GF(q^k) for q a prime, then there exists a projectivity of Type D and order k. It is now easy to construct examples showing the indispensability of the hypothesis that ϕ^{p^i} be not of Type D.

A. Let q and k be primes such that $1 + 2q \equiv 0$ modulo k. Then $n = q^k$ and $1 + n + n^2$ is divisible by k, though there exists a projectivity of Type D and order k.

B. Let $q = k = p$, a prime. Then $n = p^p$ is divisible by p, though there exists a projectivity of Type D and order p.

C*. Let q and k be primes such that $q = -1$ modulo k (e.g. $q = 19$ and $k = 5$). Then $n = q^k \equiv -1$ modulo k, though there exist projectivities of Type D and order k.

C. Let q and k be primes such that $q \equiv 1$ modulo k (e.g. $q = 11$ and $k = 5$). Then $n = q^k \equiv 1$ modulo k, though there exist projectivities of Type D and order k.

The projectivities possessing less than seven fixed points are of special interest. Note that some of their properties have been given in **1**, Corollaries 2 and 3. The projectivities without fixed points are just those of Type A; and the projectivities of Type C, possessing 1 or 3 fixed points have found special treatment in Theorems 2C′ and 2C″.

THEOREM 4. *Suppose that ϕ is a projectivity of prime power order $o(\phi) = p^m$.*

(1) *If ϕ is of* Type B *or* C, *then* $N(\phi) \equiv 1, 3$ modulo p.

(2) $N(\phi) \neq 2$.

(3) *If $N(\phi) = 3$, and if ϕ is of* Type B, *then $p = 2$.*

(4) *If $N(\phi) = 4$, then $p = 3$ and ϕ is of* Type B.

(5) *If $N(\phi) = 5$, then $p = 2$.*

(6) *If $N(\phi) = 6$, then either ϕ is of* Type B *and $p = 5$ or else ϕ is of* Type C *and $p = 3$.*

Proof. It is a consequence of Theorem 1, (a) that $N(\phi) \equiv 1 + n + n^2$ modulo p; and (1) is an immediate consequence of Theorems 2B, 2C′ and 2C″.

If $0 < N(\phi) < 7$, then ϕ is of Types B or C, since every projective plane contains at least 7 points. Thus we may make use of the statement (1) during the remainder of this proof.

(2) is a consequence of the fact that $2 \neq 1, 3$ modulo p.

If $N(\phi) = 3$ and ϕ is of Type B, then we deduce from Theorem 1, (a) and Theorem 2B that

$$3 \equiv 1 + n + n^2 \equiv 1 \text{ modulo } p,$$

implying $p = 2$.

If $N(\phi) = 4$, then ϕ is not of Type C″. If ϕ were of Type C′, then we would infer from Theorem 1, (a) and Theorem 2C′ (a) that

$$4 \equiv 1 + n + n^2 \equiv 3 \text{ modulo } p$$

which is impossible. Hence ϕ is of Type B; and it follows from Theorem 2B that

$$4 \equiv N(\phi) \equiv 1 + n + n^2 \equiv 1 \text{ modulo } p.$$

Hence $p = 3$, completing the proof of (4).

If $N(\phi) = 5$, then it follows from (1) that $5 \equiv 1, 3$ modulo p, implying $p = 2$.

If $N(\phi) = 6$, and if ϕ is of Type B, then we deduce from Theorem 2B that $6 \equiv 1 + n + n^2 \equiv 1$ modulo p or $p = 5$; and if ϕ is of Type C, then ϕ is of Type C′, and it follows from Theorem 2C′ that $6 \equiv 1 + n + n^2 \equiv 3$ modulo p or $p = 3$, completing the proof.

We conclude this section by giving a little information concerning the projectivities of Type D whose order is a power of a prime.

THEOREM 2D. *Suppose that ϕ is of prime power order $o(\phi) = p^m$, and that ϕ^{p^i} is of* Type (D, j).

(a) $p \neq 3$ *implies* $n \equiv j$ modulo p^{i+1}.

(b) $N(\phi) \neq 0$ *implies* $n \equiv j$ modulo p^{i+1}.

(c) *If* $p = 3$ *and* $N(\phi) = 0$, *then* ϕ^3 *is of* Types C″ *or* D *and* $n \equiv j$ modulo 3^i.

Proof. Since ϕ^{p^i} is of Type (D, j), it possesses exactly $1 + j + j^2$ fixed points; and it follows from Theorem 1, (a) that

$$1 + j + j^2 \equiv N(\phi^{p^i}) \equiv 1 + n + n^2 \text{ modulo } p^{i+1}$$

or $(n - j)(n + j + 1) \equiv 0$ modulo p^{i+1}. Consequently we have:

(d) $n + j + 1 \not\equiv 0$ modulo p *implies* $n \equiv j$ modulo p^{i+1}.

Suppose, next, that $N(\phi) \neq 0$. Then there exists a fixed line w of ϕ. Clearly w is a fixed line of ϕ^{p^i} which carries exactly $j + 1$ fixed points of ϕ^{p^i}. Thus it follows from Theorem 1, (b) that $j + 1 \equiv N(\phi^{p^i}, w) \equiv n + 1$ modulo p^{i+1}; and this implies (b).

Suppose, next, that $N(\phi) = 0$ and that $n + j + 1 \equiv 0$ modulo p. Then we infer from Theorem 1, (a) that

$$1 + n + n^2 \equiv N(\phi) \equiv 0 \equiv 1 + n + j \text{ modulo } p$$

or $j \equiv n^2$ modulo p. If h is a fixed line of ϕ^{p^i}, then it follows from Theorem 1, (b) that

$$j + 1 \equiv N(\phi^{p^i}, h) \equiv n + 1 \text{ modulo } p$$

or

(e) $j \equiv n$ modulo p.

But we have shown before that $n^2 \equiv j$ modulo p. Hence $n^2 \equiv n$ modulo p or $n \equiv 0, 1$ modulo p. Since we verified that $0 \equiv 1 + n + n^2$ modulo p, $n \equiv 0$ modulo p is ruled out; and we find that

(f) $n \equiv 1$ modulo p.

Consequently $0 \equiv 1 + n + n^2 \equiv 3$ modulo p, proving $p = 3$. This completes the proof of (a).

Since $N(\phi) = 0$ and $N(\phi^{3^i}) \neq 0$, there exists a smallest integer k such that $N(\phi^{3^k}) \neq 0$. Clearly $0 < k$ and $N(\phi^{3^{k-1}}) = 0$.

Case 1. ϕ^{3^k} is of Type D.

Then there exists an integer j', not less than 2, such that ϕ^{3^k} is of Type (D, j'); and it follows from (e), (f) that $1 \equiv n \equiv j'$ modulo 3. It follows from **2**, Theorem 1, (a) that

$$0 \equiv N(\phi^{3^{k-1}}) \equiv N(\phi^{3^k}) \equiv 1 + j' + j'^2 \text{ modulo } 3^k.$$

But $j' = 1 + 3j''$ where $0 < j''$, since otherwise $j' < 2$. Hence

$$0 \equiv 1 + j' + j'^2 \equiv 3 + 9j''(1 + j'') \text{ modulo } 3^k.$$

From $3 \not\equiv 0$ modulo 9 we infer $k = 1$, so that in this case ϕ^3 is of Type D.

Case 2. ϕ^{3^k} is not of Type D.

If ϕ^{3^k} were of Types B or C$'$, then we would infer from **1**, Theorem 2 the existence of fixed elements of ϕ which is impossible. Since $N(\phi^{3^k}) \neq 0$, it follows now that ϕ^{3^k} is of Type C$''$; and it follows from Theorem 2C$''$, (c) that $N(\phi^3) = 3$, proving that in this case too $k = 1$ and that ϕ^3 is of Type C$''$.

Thus we have shown in both cases that $N(\phi^3) \neq 0$; and hence it follows from (b), applied upon ϕ^3, that $n \equiv j$ modulo p^i, completing the proof.

4. Special groups of projectivities. If Δ is a group of projectivities,

then we denote by $o(\Delta)$ the order of the group Δ. Furthermore we say that an element x in the projective plane Π is Δ-invariant, if it is left invariant by every projectivity in Δ. (See **1**, Remark 4.)

DEFINITION. *The group Δ of projectivities of the projective plane Π is a special group of projectivities, if the identity is the only projectivity of* Type D *in Δ.*

If Π is the projective plane over a Galois Field, then its groups of special collineations are special groups of projectivities in the meaning of the above Definition (see **3**, Remark 5). The converse of this statement is, however, not true, as may be seen from simple examples.

THEOREM 1. *Suppose that Δ is a special group of projectivities. Then*

(a) *$o(\Delta)$ is a divisor of $(1+n+n^2)(1+n)n^3(n-1)^2$.*

(b) *If there exist Δ-invariant points, then $o(\Delta)$ is a divisor of $(1+n)n^3(n-1)^2$.*

(c) *If there exist different Δ-invariant points, then $o(\Delta)$ is a divisor of $n^2(n-1)^2$.*

(d) *If the Δ-invariant points are not collinear, then $o(\Delta)$ is a divisor of $(n-1)^2$.*

Proof.[16] The ordered quadruplet of points (R, S, T, U) will be termed an ordinary quadrangle, if no three of the four points R, S, T, U are collinear. The set H of ordinary quadrangles admits Δ, if $(R\phi, S\phi, T\phi, U\phi)$ is in H, whenever (R, S, T, U) is in H and ϕ is in Δ. We prove:

(i) *If the set* H *of ordinary quadrangles admits the special group Δ of projectivities, then $o(\Delta)$ is a divisor of the number of quadrangles in* H.

Suppose that ϕ is a projectivity in Δ and (M, N, P, Q) an ordinary quadrangle in H such that $(M, N, P, Q) = (M\phi, N\phi, P\phi, Q\phi)$. Then ϕ is of Type D, as has already been pointed out when introducing, in Section **1**, the types. But Δ is special and thus it follows that $\phi = 1$. Consequently every quadrangle in H is mapped by the projectivities in Δ upon exactly $o(\Delta)$ distinct quadrangles in H and these sets are mutually exclusive, proving (i).

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[16] The author is much indebted to the referee for pointing out this elegant proof which is much simpler than the author's original one.

Suppose that the Δ-invariant points are not collinear. Then there exist three points P, Q, R which are not collinear and which are Δ-invariant. Consider the set $[P, Q, R]$ of all the ordinary quadrangles of the form (P, Q, R, X). This set contains $(n-1)^2$ quadrangles, since there exist just $(n-1)^2$ points in Π which are neither on $P+Q$ nor on $Q+R$ nor on $R+P$—there are just $3n$ points on these three lines. This set $[P, Q, R]$ admits Δ. Hence it follows from (i) that $o(\Delta)$ is a divisor of $(n-1)^2$, proving (d).

Assume, next, that there exist two different Δ-invariant points, say $P \neq Q$. Consider the set $[P, Q]$ of all the ordinary quadrangles of the form (P, Q, X, Y). This set contains $n^2(n-1)^2$ quadrangles, since X may be selected as any one of the n^2 points, not on $P+Q$; and $[P, Q]$ admits Δ. As before we deduce from (i) that $o(\Delta)$ is a divisor of $n^2(n-1)^2$, proving (c).

Assume now that there exists at least one Δ-invariant point, say P. Consider the set $[P]$ of all the ordinary quadrangles of the form (P, X, Y, Z). This set contains exactly $(n+n^2)n^2(n-1)^2$ quadrangles, since X may be selected as anyone of the $n+n^2$ points, not P. But $[P]$ admits Δ; and thus (b) is a consequence of (i).

Consider, finally, the set of all the ordinary quadrangles (W, X, Y, Z). Since W may be selected in $1+n+n^2$ different fashions, this set contains $(1+n+n^2)(1+n)n^3(n-1)^2$ different quadrangles. Since the set of all ordinary quadrangles admits Δ, our contention (a) is a consequence of (i), completing the proof.

THEOREM 2A. *Suppose that Δ is a special group of projectivities and that 3 is not a common divisor of $o(\Delta)$ and $1+n+n^2$. Then every projectivity, not 1, in Δ is fixed element free if, and only if, $o(\Delta)$ is a divisor of $1+n+n^2$.*

Proof. If the projectivities, not 1, in Δ are fixed element free, then every point P is mapped by the projectivities in Δ upon $o(\Delta)$ different points. Since these sets $P\Delta$ are mutually exclusive, it follows that $o(\Delta)$ is a divisor of the number of $1+n+n^2$ of points.

Suppose, conversely, that $o(\Delta)$ is a divisor of $1+n+n^2$ and that $\phi \neq 1$ is a projectivity in Δ. Then there exists an integer k such that ϕ^k is of order a prime number p. Since ϕ^k is in Δ, it is not of Type D and its order p is a divisor of $o(\Delta)$ and therefore of $1+n+n^2$. Hence $p \neq 3$; and it follows from **3**, Theorem 3, (A) that ϕ^k is fixed element free. Consequently ϕ itself is fixed element free, completing the proof.

Whenever we speak of p-groups, this shall signify that p is a prime number and that the group under consideration is of order a power of p. The following proposition will be applied several times.

LEMMA 1. *If the group Δ of projectivities of Π is a p-group, if the set Ξ of elements in Π is mapped upon itself by the projectivities in Δ, then the number of Δ-invariant elements in Ξ is,* modulo p, *congruent to the number of elements in Ξ.*

Proof. If x is an element in Ξ, then consider the set $x\Delta$ of all the elements in Π upon which x is mapped by projectivities in Δ. Every set $x\Delta$, for x in Ξ, is part of Ξ; and these sets are mutually exclusive. If x is Δ-invariant, then $x\Delta$ consists of one and only one element, namely x. If x is not Δ-invariant, then the number of elements in $x\Delta$ is a multiple of p—as a matter of fact a positive power of p. Our contention is an immediate consequence of these facts.

THEOREM 2B. *Suppose that Δ is a special group of projectivities.*

(a) *Every projectivity, not* 1, *in Δ is of* Type B *if, and only if, $o(\Delta)$ is a divisor of n^3.*

(b) *If Δ is a p-group and p a divisor of n, then there exist Δ-invariant points (lines) and Δ-invariant points on every Δ-invariant line.*

Proof. Suppose first that every projectivity, not 1, in Δ is of Type B. If $o(\Delta)$ is divisible by the prime number p, then there exists, by Cauchy's Theorem, a projectivity ϕ of order p in Δ. Since ϕ is of Type B, we infer from **3**, Theorem 3, (B), that p is a divisor of n. Thus every prime divisor of $o(\Delta)$ is prime to $(1+n+n^2)(n^2-1)$; and it follows from Theorem 1, (a) that $o(\Delta)$ is a divisor of n^3.

Assume, conversely, that $o(\Delta)$ is a divisor of n^3. If $\phi \neq 1$ is a projectivity in Δ, then there exists a positive integer k such that ϕ^k is of order a prime number p. Since ϕ^k is in the special group Δ, it is not of Type D; and hence it follows from **3**, Theorem 3, (B), that ϕ^k is of Type B as p is a divisor of $o(\Delta)$ and therefore of n. It follows now from **1**, Corollary 1 that ϕ itself is of Type B, completing the proof of (a).

If p is a divisor of n, then $1+n+n^2 \equiv 1$ modulo p. Thus it follows from Lemma 1, applied on the set of all the $1+n+n^2$ points (lines), that the number of Δ-invariant points (lines) is congruent to 1 modulo p, proving

the first part of (b). If h is a Δ-invariant line, then it follows from Lemma 1, applied on the set of all the $n+1$ points on h that the number of Δ-invariant points on h is congruent to $n+1 \equiv 1$ modulo p, proving the existence of Δ-invariant points on h. This completes the proof.

Remark 1. It is clear from the proof of (a)—and may be deduced from Theorem 1, (a)—that $o(\Delta)$ is a divisor of n^3, if each prime divisor of $o(\Delta)$ is a divisor of n.

COROLLARY 1. *If every projectivity, not* 1, *in* Δ, *is of* Type B, *and if there exist at least two* Δ*-invariant points (lines), then* $o(\Delta)$ *is a divisor of* n^2 *and all the projectivities, not* 1, *in* Δ *have the same axis (center).*

Proof. It is a consequence of Theorem 1, (b) and Theorem 2B, (a) that $o(\Delta)$ is a common divisor of n^3 and $n^2(n-1)^2$. Hence $o(\Delta)$ is a divisor of n^2. The second contention is a consequence of the fact that the axis is the only fixed line of a projectivity of Type B which carries more than one fixed point.

In analogy to the division of cases used in **3**, Theorem 3, we introduce now the following *notation*: the projectivity ϕ is said to be of *Type C**, whenever it is of Type C and $N(\phi)=1$; and it is said to be of *Type C***, whenever it is of Type C though $N(\phi) \neq 1$.

THEOREM 2C*. *Suppose that* Δ *is a special group of projectivities and that* $o(\Delta)$ *and* $n+1$ *are not both even.*

(a) *Every projectivity, not* 1, *in* Δ *is of* Type C* *if, and only if,* $o(\Delta)$ *is a divisor of* $n+1$.

(b) *If* Δ *is nilpotent and* $o(\Delta)$ *a divisor of* $n+1$, *then all the projectivities, not* 1, *in* Δ *have the same fixed point and the same fixed line.*

Proof. Suppose first that every projectivity, not 1, in Δ is of Type C*. Thus every projectivity $\phi \neq 1$ in Δ possesses one and only one fixed point $H(\phi)$, one and only one fixed line $h(\phi)$; and $H(\phi)$ is not on $h(\phi)$. Assume now that the prime number p is a divisor of $o(\Delta)$. There exists, by Cauchy's Theorem, a projectivity ϕ of order p in Δ. If p were 2, then it would follow from our hypothesis that n would be even; and it would follow from **3**, Theorem 3, (B), that ϕ would be of Type B, an impossibility. Thus p is odd. But ϕ is of Type C*; and hence it follows from **3**, Theorem 3, (C*),

that p is a divisor of $n+1$. Consequently every prime divisor of $o(\Delta)$ is prime to $(1+n+n^2)n(n-1)$; and it follows from Theorem 1, (a) that $o(\Delta)$ is a divisor of $n+1$.

Assume, conversely, that $o(\Delta)$ is a divisor of $n+1$. If $\phi \neq 1$ is a projectivity in Δ, then there exists a positive integer k such that ϕ^k is of order a prime number p. Since ϕ^k is in the special group Δ, it is not of Type D; since p is, as a divisor of $o(\Delta)$, a divisor of $n+1$, it follows from our hypothesis that $p \neq 2$ and from **3**, Theorem 3, (C*), that ϕ^k is of Type C*. Hence it follows from **1**, Theorem 2—and the fact that fixed elements of ϕ are also fixed elements of ϕ^k—that ϕ itself is of Type C*, completing the proof of (a).

If Δ is nilpotent, then Δ is the direct product of p-groups Δ_p. If $o(\Delta)$ is a divisor of $n+1$, and if $\Delta_p \neq 1$, then the prime number p is a divisor of $n+1$. This implies that $1+n+n^2 \equiv 1$ modulo p; and we infer from Lemma 1, applied on the set of all the $1+n+n^2$ points, the existence of Δ_p-invariant points and lines. Since it follows from (a) that all the projectivities, not 1, in Δ are of Type C*, we infer the existence of one and only one common fixed point $P(p)$ (fixed line $h(p)$) of the projectivities, not 1, in Δ_p. If p and q are different prime numbers, and if Δ_p and Δ_q are both different from 1, then consider projectivities ϕ and γ of orders p and q respectively. Since $\phi\gamma = \gamma\phi$, it follows that both ϕ and γ are powers of $\phi\gamma$. But $\phi\gamma$ is in Δ and therefore of Type C*. It follows from **1**, Theorem 2, that the only fixed point $P(p)$ of ϕ is the only fixed point of $\phi\gamma$; and the only fixed point $P(q)$ of γ is likewise the only fixed point of $\phi\gamma$. Hence $P(p) = P(q)$; and $h(p) = h(q)$ is seen in a similar fashion. From this fact our contention (b) is now easily deduced.

Remark 2. It is clear from the proof of (a)—and may be deduced from Theorem 1, (a)—that $o(\Delta)$ is a divisor of $n+1$, if it is odd and each of its prime divisors is a divisor of $n+1$.

THEOREM 2C**. *Suppose that Δ is a special p-group not* 1, *and that $p \neq 2, 3$. Then the following properties imply each other.*

(a) *The Δ-invariant points are not collinear.*

(b) *Every projectivity, not* 1, *in Δ is of* Type C**.

(c) *p is a divisor of $n-1$.*

(d) *$o(\Delta)$ is a divisor of $(n-1)^2$.*

Proof. If (a) is satisfied, and if $\phi \neq 1$, then there exist at least three non-collinear fixed points of ϕ. Since ϕ cannot be of Type D, as Δ is special, this implies that ϕ is of Type C**. Thus (a) implies (b). That (b) implies (c), may be inferred from **3**, Theorem 3, (C); if (c) is satisfied, then p is prime to $(1+n+n^2)n^3(n+1)$, since $p \neq 2, 3$; and (d) is a consequence of Theorem 1, (a).

Assume finally the validity of (d). Then p is a divisor of $n-1$ and hence $1+n+n^2 \equiv 3$ modulo p. Since $p \neq 2, 3$, we may infer from Lemma 1 the existence of at least three different Δ-invariant points. If P and Q are two different Δ-invariant points, then there exist $n^2 \equiv 1$ modulo p points outside the line $P+Q$; and hence it follows from Lemma 1 that there exist Δ-invariant points, not on $P+Q$, proving the validity of (a) and completing the proof.

THEOREM 2C. *If Δ is a special group of projectivities whose order is prime to 3, then the following properties imply each other.*

(a) *$o(\Delta)$ is a divisor of $(n+1)(n-1)^2$.*

(b) *Every prime divisor of $o(\Delta)$ is a divisor of n^2-1.*

(c) *Every projectivity, not 1, in Δ is of* Type C.

Proof. It is obvious that (a) implies (b). If (b) is satisfied, then $o(\Delta)$ is prime to $(1+n+n^2)n^3$, since $o(\Delta)$ is prime to 3. But Δ is special. Hence it follows from Theorem 1, (a) that $o(\Delta)$ is a divisor of $(n+1)(n-1)^2$, showing that (a) and (b) are equivalent.

Assume now the validity of (c). If p is a prime divisor of $o(\Delta)$, then there exists, by Cauchy's Theorem, a projectivity ϕ of order p in Δ. Since $p \neq 3$, and since ϕ is, by hypothesis, of Type C′ or C″, we infer from **3**, Theorem 2C′ and **3**, Theorem 2C″, that p is a divisor of n^2-1, showing that (b) is a consequence of (c).

Assume, finally, the validity of (b). If ϕ is a projectivity, not 1, in Δ, then there exists a positive integer k such that ϕ^k is of order a prime number p. Since p is a prime divisor of $o(\Delta)$, p is different from 3 and p is a divisor of n^2-1. Since Δ is special and $\phi^k \neq 1$, it follows that ϕ^k is not of Type D. It follows from **3**, Theorem 2A and **3**, Theorem 2B, that ϕ^k is neither of Type A nor of Type B. Consequently ϕ^k is of Type C. But the order of ϕ is prime to 3, since the order of Δ is prime to 3. Hence it follows from **1**, Corollary 4 that ϕ itself is of Type C, showing that (c) is a consequence of (b). This completes the proof.

COROLLARY 2. *If Δ is a special group of projectivities whose order is prime to 3, then the following properties imply each other.*

(a) *$o(\Delta)$ is a divisor of $(n+1)n^3(n-1)^2$.*

(b) *Every prime divisor of $o(\Delta)$ is a divisor of $n(n^2-1)$.*

(c) *Every projectivity in Δ possesses fixed elements.*

Proof. It is clear that (a) implies (b). If (b) is valid, then $o(\Delta)$ is prime to $1+n+n^2$, since it is prime to 3. But Δ is special; and hence it follows from Theorem 1, (a) that $o(\Delta)$ is a divisor of $(n+1)n^3(n-1)^2$, showing the equivalence of (a) and (b).

Assume now the validity of (c). If p is a prime divisor of $o(\Delta)$, then there exists, by Cauchy's Theorem, a projectivity ϕ of order p in Δ. Since $p \neq 3$, and since ϕ is, by hypothesis, NOT fixed element free, it follows from **3**, Theorem 3, (A), that p is prime to $1+n+n^2$. Hence it follows from **3**, Corollary 7, that p is a divisor of $(n-1)n(n+1)$, showing that (b) is a consequence of (c).

Assume, finally, the validity of (b). If ϕ is a projectivity, not 1, in Δ, then we distinguish two cases.

Case 1. n and $o(\phi)$ are relatively prime.

If p is a prime divisor of $o(\phi)$, then p is a divisor of $o(\Delta)$ so that $p \neq 3$ is a divisor of n^2-1. But $o(\phi)$ is the order of the cyclic group generated by ϕ which group is special as a subgroup of Δ. Hence we may infer from Theorem 2C that ϕ is of Type C so that in particular ϕ possesses fixed elements.

Case 2. n and $o(\phi)$ are *not* relatively prime.

Then there exists a prime number q which is a common divisor of n and $o(\phi)$. There exists furthermore a positive integer i such that ϕ^i is of order q. Since ϕ^i, as an element in the special group Δ, is not of Type D, it follows from **3**, Theorem ~~2~~3B, that ϕ^i is of Type B. Then it follows from **1**, Theorem 2, that center and axis of ϕ^i are fixed elements of ϕ.

Thus we have shown in all generality that (c) is a consequence of (b) completing the proof.

5. The group of special collineations. If Π is the projective plane over the Galois Field GF(n), then we have pointed out in **3**, Remark 5, that the

group Z of special collineations of Π is a special group of projectivities whose order is $(1+n+n^2)(1+n)n^3(1-n)^2$. This shows the impossibility of improving the limit given in Theorem 1, (a). §4

If P is some point in Π and $Z(P)$ the group of all the projectivities in Z which leave invariant the point P, then the order of $Z(P)$ is $(1+n)n^3(1-n)^2$, showing the impossibility of improving the limit given in Theorem 1, (b) and Corollary 2, (a). But there does not exist any $Z(P)$-invariant line, showing the impossibility of substituting in Corollary 2 for (c) the condition: "there exist Δ-invariant lines."

If P and Q are different points in Π and $Z(P,Q)$ the group of all the projectivities in Z which leave invariant P and Q, then the order of $Z(P,Q)$ is $n^2(n-1)$; but the only $Z(P,Q)$-invariant elements are P, Q and $P+Q$, showing the impossibility of improving Theorem 1, (c).

If the points P, Q, R in Π are not collinear, and if $Z(P,Q,R)$ is the group of all the projectivities in Z which leave invariant P, Q and R, then this group is of order $(n-1)^2$, showing the impossibility of improving the limits given in Theorem 1, (d) and Theorem 2C**.

Singer (1) has shown that Z possesses a cyclic subgroup of order $1+n+n^2$ whose generator effects a cyclic permutation of the $1+n+n^2$ points in Π. Thus Theorem 2A gives a best limit. The method employed by Singer (1) may be used to construct a cyclic subgroup of order $n+1$ of Z, whose generator leaves invariant a point P and effects a cyclic permutation of the $n+1$ points on some line, not through P, showing the impossibility of improving the limit given in Theorem 2C*.

Consider finally the subgroup Γ of Z whose elements may be represented by matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}.$$

It is clear that Γ is of order n^3 and that the projectivities in Γ do not all have the same axis and/or center. Thus the limits given in Theorem 2B cannot be improved.

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THE SUM FORMULA OF EULER-MACLAURIN AND THE INVERSIONS OF FOURIER AND MÖBIUS.*

By Aurel Wintner.

1. By further developing the considerations of [11], the present paper deals with the legitimacy of equidistant Riemannian evaluations of improper integrals and with its connections with both kinds of inversions mentioned in the title.

The connection with the second of these inversions, that of Möbius, can easily be hidden. Actually, it is the basis of the following theorem:

(α) *If $f(x)$ is continuous on the half-open interval $0 < x \leqq 1$, and if the limit*

$$\lim_{\epsilon \to 0} \epsilon \sum_{n\epsilon \leqq 1} f(n\epsilon) \tag{1}$$

exists, then the improper integral

$$\int_{+0}^{1} f(x)dx \tag{2}$$

is convergent (and has the same value as (1)).

This theorem, (α), sounds innocent enough. Actually, it contains the following theorem of Hardy and Littlewood [5]:

(α_0) *Every series summable in Lambert's sense (L) is summable in Abel's sense (A).*

That (α) is quite deep, follows from the implication (α) $\to$ (α_0) and from the fact that, according to Hardy and Littlewood, (α_0) contains the prime number theorem (incidentally, the converse does not hold, since the proof depends on a refinement of the prime number theorem, that is, on the non-vanishing of $\zeta(s)$ on a certain open domain containing the line, $\sigma = 1$, of the prime number theorem).

In order to verify that (α) $\to$ (α_0), the continuity of $f(x)$, assumed in (α), is sufficient. From quite another point of view, namely, from that of the heavy discontinuities compatible with the measurability of a function, it is of interest that (α) can be refined as follows:

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(α^*) *If $f(x)$ is L-integrable on every interval $\epsilon \leqq x \leqq 1$, where $\epsilon > 0$, and if the limit* (1) *exists, then the improper integral* (2) *is convergent and has the value* (1).

The converse of (α) is obviously false (for a simple example, which can readily be refined, cf. [2], p. 230). That even the converse of the corollary, (α_0), of (α) is false, is shown by an arithmetical example of Hardy and Littlewood [6], pp. 265-269. What is true in the "converse" direction proves to be a criterion of the following type:

(β) *If $f(x)$ is of bounded variation on every interval $\epsilon \leqq x \leqq 1$, where $\epsilon > 0$, and behaves, as $x \to 0$, so as to satisfy the restriction*

$$\int_{\epsilon}^{1} | df(x) | = o(\epsilon^{-1}), \tag{3}$$

then the convergence of the improper integral (2) *implies that the limit* (1) *exists (and has the same value as* (2)).

Corresponding to the implication $(\alpha) \to (\alpha_0)$, the last criterion, (β), implies the following (partial) converse of (α_0):

(β_0) *Every absolutely* (A)*-summable series is* (L)*-summable.*

Not even this corollary of (β) seems to occur in the literature. What is known is that particular case of (β_0) in which the series is absolutely (A)-summable for the trivial reason that the derivative of the Abelian generator of the series has a *monotone majorant* which is integrable (criterion of Ananda-Rao [1]; a simplified proof is given by Hardy and Littlewood [6], pp. 258-259). In fact, the situation is as follows:

A series Σa_n is called (A)-summable if $A(r) = \Sigma a_n r^n$ (is convergent when $r < 1$ and) tends to a limit, $A(1-0)$, as $r \to 1$. Clearly, this is equivalent to the convergence of the improper integral

$$\int_{0}^{1-0} A'(r)dr, \qquad \text{where} \qquad A' = dA/dr.$$

Correspondingly, J. M. Whittaker has called the series Σa_n absolutely (A)-summable if this integral is absolutely convergent, that is, if

$$\int_{0}^{1-0} | dA(r) | < \infty.$$

What the known particular case of (β_0), referred to above, states is that $\Sigma\, a_n$ is (L)-summable if the condition required by the last formula line is replaced by the assumption

$$\int_0^{1-0} (\max_{0 \leqq q \leqq r} |A'(q)|)\, dr < \infty,$$

which, of course, is more strict than

$$(4) \qquad \int_0^{1-0} |A'(r)|\, dr < \infty,$$

the absolute (A)-summability of $\Sigma\, a_n$.

Only the second of the inversions mentioned in the title, the inversion of Möbius, seems to be involved above. That the other inversion, that concerning Fourier transforms, can be involved just as well, is clear from the connections considered in [11]. In fact, the connection between the inversion formulae of Möbius and Fourier becomes manifest if the sum occurring in (1), a sum arrested at $x = 1$, is made an infinite sum. The latter is an Euler-Maclaurin expression and admits, therefore, of Poisson's Fourier analysis (in [11], only the arrested sums, leading to Fourier constants rather than to Fourier transforms, have been considered).

This Fourier analysis can be carried out very simply, by starting from a suitable formulation of the Euler-Maclaurin formula itself. In fact, the latter leads automatically to the function $x - [x]$, where $[x]$ denotes the greatest integer not exceeding x. All that will then be needed is the insertion of the Fourier series of the periodic function $x - [x]$ into the Euler-Maclaurin formula. What will be needed of the properties of this series,

$$(5) \qquad \tfrac{1}{2} - \pi^{-1} \sum_{n=1}^{\infty} n^{-1} \sin 2\pi n x,$$

is that (5) is convergent (uniformly on every closed interval not containing an $x = [x]$); that

(6) the sum of (5) is $x - [x]$ if $x \neq [x]$, and 0 if $x = [x]$;

finally that

(7) the partial sums of (5) are uniformly bounded.

2. The whole theory centers about the identity claimed by the following remark:

(i) *If $f(x)$ is of bounded variation on the half-line $x \geqq 1$,*

(8) $$\int_1^\infty |df(x)| < \infty,$$

and if the value of $\lim_{x\to\infty} f(x)$ (which then exists) is

(9) $$f(\infty) = 0,$$

then the limit

(10) $$\lim_{x\to\infty} \left(\sum_{n\leqq x} f(n) - \int_1^x f(u)du \right),$$

where $n = 1, 2, \cdots$, exists and is equal to the value of

(11) $$\int_1^\infty (x - [x])df(x).$$

If $f(x)$ is discontinuous at an integer, $x = n$, then both factors occurring in the Stieltjes integral (11) are discontinuous at $x = n$, and so (11) must then be meant in the Lebesgue-Stieltjes sense, rather than as an improper Riemann-Stieltjes integral.

The actual content of (i) is just the Euler-Maclaurin sum formula, except that the above formulation avoids the usual conditions of smoothness. In fact, the usual wording presupposes not only the continuity of $f(x)$ but its absolute continuity as well; cf. [4]. The latter restriction would preclude the applicability of the Euler-Maclaurin rule to "explicit" cases of considerable interest. Actually, the proof becomes particularly simply precisely under the above general asumptions.

In fact, since $[x] - x = O(1)$ as $x \to \infty$, and since (9) means that $f(x) = o(1)$, a partial integration gives

(12) $$\int_1^x f(u)d([u] - u) = o(1)O(1) - 0 - \int_1^x ([u] - u)df(u).$$

But $[u] - u = O(1)$ and (8) imply that, as $x \to \infty$, the expression on the right of (12) tends to the value (11), whereas the integral on the left of (12) is identical with the difference the limit of which is taken in (10). This proves (i).

(ii) *If $f(x)$ is defined for $x > 1$ in such a way that the Lebesgue-Stieltjes integral of $u - [u]$ with respect to $f(u)$ exists on every finite*

interval $1 \leqq u \leqq x$ *and tends, as* $x \to \infty$, *to a finite limit,* (11), *and if* $f(x) \to 0$, *then the limit* (10) *exists and equals the value of* (11).

This generalization of (i) is clear from the proof of (i). In fact, (12) can be applied under the present assumptions also.

An immediate consequence of (i) is the following fact:

(iii) *If* $f(x)$ *is of bounded variation on every closed half-line contained in the open half-line* $x > 0$, *and if the improper integral of* $f(x)$ *on such a closed half-line is convergent (possibly just conditionally), then the series* $\Sigma f(\epsilon n)$ *is convergent and satisfies the inequality*

$$(13) \qquad \left| \epsilon \sum_{n=1}^{\infty} f(\epsilon n) - \int_{\epsilon}^{\infty} f(x)dx \right| \leqq \epsilon \int_{\epsilon}^{\infty} | df(x) |,$$

where $\epsilon > 0$ *is arbitrary.*

Easy examples show that the convergence of *both*

$$(14) \qquad \int_1^{\infty} | df(x) | \quad \text{and} \quad \int_1^{\infty} f(x)dx$$

does not preclude the divergence of

$$(15) \qquad \int_1^{\infty} | f(x) | dx;$$

so that the parenthetical remark preceding (13) is not illusory.

Since the convergence of the first of the integrals (14) implies the existence of a limit $f(\infty)$, and since the non-vanishing of the latter is prevented by the convergence of the second of the integrals (14), the assumptions of (iii) contain those of (i). In addition, (i) is now applicable to $f(\epsilon x)$ instead of to $f(x)$, if $\epsilon > 0$ is arbitrarily fixed. Thus, (10) being equal to (11),

$$(16) \qquad \lim_{x\to\infty} \Big(\sum_{n \leqq x/\epsilon} f(\epsilon n) - \int_1^{x/\epsilon} f(\epsilon u)du \Big) = \int_1^{\infty} (x - [x]) d_x f(\epsilon x).$$

If this is multiplied by ϵ, then, since the second of the integrals (14) is convergent, what results on the left of (16) is the difference the absolute value of which is taken on the left of (13). Since the integral on the right of (16) is majorized by

$$\int_1^\infty (x - [x]) \, | d_x f(\epsilon x) | \leqq \int_1^\infty | d_x f(\epsilon x) | = \int_\epsilon^\infty | df(x) | ,$$

the assertion, (13), of (iii) follows.

(iv) *If both integrals* (14) *are convergent and*

$$\int_\epsilon^1 | df(x) | = o(\epsilon^{-1}) \tag{17}$$

as $\epsilon \to 0$, *then*

$$\epsilon \sum_{n=1}^\infty f(n\epsilon) - \int_\epsilon^\infty f(x) dx \to 0. \tag{18}$$

This is a corollary of (iii). It should be noted that neither of the terms on the left of (18) need tend to a limit. All that follows is that, if one of these terms tends to a limit, the other must tend to the same limit. Thus (iv) implies that *the Riemannian equidistant evaluation* of a convergent (doubly improper) integral

$$\int_0^\infty f(x) dx = \int_0^1 + \int_1^\infty, \quad \text{where} \int_0^1 = \lim_{\epsilon \to 0} \int_\epsilon^1, \quad \int_1^\infty = \lim_{x \to \infty} \int_1 \tag{19}$$

is legitimate whenever

$$\int_\epsilon^\infty | df(x) | = o(\epsilon^{-1}). \tag{20}$$

(v) *If* $f(x)$ *has on every half-line* $x > \epsilon$ *a finite total variation which satisfies* (20) *as* $\epsilon \to 0$, *and if both improper integrals occurring in* (19) *are convergent* (*possibly just conditionally*), *then*

$$\epsilon \sum_{n=1}^\infty f(n\epsilon) \to \int_0^\infty f(x) dx. \tag{21}$$

If $f(x)$ is chosen to be 0 when $x > 1$, and if the continuous variable ϵ is replaced by the reciprocal value of an integer, it follows that, for every function $f(x)$ which is defined on the interval $0 < x < 1$ so as to satisfy (17), the (equidistant) Riemannian relation

$$\sum_{k=1}^{n} f(k/n)/n \to \lim_{\epsilon\to 0} \int_{\epsilon}^{1} f(x)\,dx \qquad (n\to\infty) \tag{22}$$

holds whenever the improper integral on the right of (21) is convergent. This corollary of (v) is sharper than the usual criterion for the truth of (22); a criterion which replaces (17) by the monotony of $f(x)$ (cf. [2], pp. 229-230). For, on the one hand, (17) and the convergence of the improper integral on the right of (21) do not imply the absolute convergence of that integral. And, on the other hand, the convergence of the latter and the monotony of $f(x)$ imply the absolute convergence of the integral, which in turn implies, again by the monotony of $f(x)$, that

$$\epsilon f(\epsilon) \to 0 \text{ as } \epsilon \to 0, \tag{23}$$

and so (17) follows by necessity.

The criteria (iii), (iv), (v) do not take care of cases exemplified by the function $f(x) = (\sin x)/x$. Functions of this type are included in the following theorem:

(vi) *If λ is a non-vanishing real number, then the limit relation*

$$\epsilon \sum_{n=1}^{\infty} f(n\epsilon) e^{i\lambda n\epsilon} \to \int_{0}^{\infty} f(x) e^{i\lambda x} dx \qquad (\epsilon \to 0) \tag{24}$$

holds whenever

$$\int_{0}^{\infty} |\,df(x)| < \infty \qquad \text{and } f(\infty) = 0 \tag{25}$$

(the convergence of the series and of the integral occurring in (24) being implied by (25), since $\lambda \neq 0$).

The truth of the parenthetical remark following (25) is seen after a partial summation and a partial integration of the series (24) and of the integral (24), respectively.

The proof of (vi) may be omitted, since it can be read off from one given by Bromwich and Hardy [2], pp. 231-233. It is true that they assume, instead of (25), that $f(x)$ is a *monotone* function satisfying (25), and it is also true that not every real-valued $f(x)$ satisfying (25) is the sum of two monotone functions both of which satisfy (25). Nevertheless, a glance at

the proof given by Bromwich and Hardy shows that (25) alone suffices for the proof of (24).

Incidentally, the proof of (vi) differs from that of the preceding criteria only insofar as what corresponds to an application of the second mean-value theorem (namely, preliminary partial summations and partial integrations of the respective sums and integrals) must precede an application of the first mean-value theorem.

3. If

$$\mu^*(x) = \sum_{n=1}^{x} \mu(n)/n, \qquad \left(\sum_{n=1}^{x} = \sum_{n \leqq x}\right), \tag{26}$$

where $\mu(n)$ denotes Möbius' factor, then the prime number theorem is known to be equivalent to $\mu^*(x) = o(1)$, where $x \to \infty$. In their proof of the theorem denoted above by (α_0), Hardy and Littlewood [5] refer to somewhat more than $\mu^*(x) = o(1)$, namely, to $\mu^*(x) = O(\log x)^{-2}$. What is actually needed is something between these two estimates, namely,

$$\sum_{n=1}^{\infty} |\mu^*(n)|/n < \infty. \tag{27}$$

In order to prove that the theorem denoted above by (α^*) can be concluded from (27), it is sufficient to repeat the proof of (α_0). The same proof also supplies the following extension of (α^*):

(vii) *Let $f(x)$ be L-integrable on every closed half-line contained in the open half-line $x > 0$, and suppose that $f(x)$ tends quite rapidly to 0, as $x \to \infty$; for instance, so that*

$$f(x) = O(x^{-1-\eta}) \text{ as } x \to \infty \tag{28}$$

holds for some $\eta > 0$. Then the limit

$$\lim_{\epsilon \to 0} \epsilon \sum_{n=1}^{\infty} f(n\epsilon) \tag{29}$$

cannot exist unless the improper integral

$$\int_{+0}^{\infty} f(x)\,dx = \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} f(x)\,dx \tag{30}$$

is convergent, and has the same value as (29).

Needless to say, the convergence of the series occurring in (29) and of the integral occurring on the right of (30) is assured by (28) for every $\epsilon > 0$. If the convergence, or the absolute convergence, of these expressions (for every $\epsilon > 0$) is granted, one has the impression that the actual rôle of (28) is that of a "Tauberian condition." But this is not quite the case. In fact, something like (28) will be needed twice, both times at very *rough* stages of the proof of (vii); namely, in order to make Möbius' inversion legitimate at a fixed $\epsilon > 0$, that is, *before* the application of the crucial limit process, $\epsilon \to 0$, and then, in order to justify a term-by-term integration, but again just for a fixed $\epsilon > 0$. Nevertheless, (28) cannot be replaced by

$$(31) \qquad \sum_{n=1}^{\infty} | f(n\epsilon) | < \infty \text{ and } \int_{\epsilon}^{\infty} | f(x) | \, dx < \infty,$$

where $\epsilon > 0$ is arbitrary. For, if (28) could be relaxed to (31) in (vii), it would be clear from the proof of (vii) (cf. below) that not only (α_0) but also the converse of (α_0) is true. However, the converse of (α_0) is disproved by the example of Hardy and Littlewood, mentioned after (α^*). The situation is cleared up by the fact that, as proved in [11], pp. 17-18, the absolute convergence of the series (31) is insufficient for the legitimacy of Möbius' inversion (at a fixed $\epsilon > 0$).

In the proof of (vii), it can be assumed that the limit (29), the existence of which is assumed, is 0. For, if it is not 0, it can be made 0 by an alteration of $f(x)$ on an unessential range, say on the interval $1 < x < 2$. Thus the assumptions of (vii) are reduced to (28) and

$$(32) \qquad F(x) = o(x^{-1}) \text{ as } x \to 0,$$

where $x > 0$ and

$$(33) \qquad F(x) = \sum_{n=1}^{\infty} f(nx),$$

whereas the assertion of (vii) becomes

$$(34) \qquad \lim_{x \to 0} \int_{x}^{\infty} f(u) \, du = 0;$$

cf. (29) and (30).

Möbius' inversion of (33) is

$$(35) \qquad f(x) = \sum_{n=1}^{\infty} \mu(n) F(nx),$$

where $x > 0$ is arbitrary. The legitimacy of this inversion is guaranteed by (28); in fact, somewhat less than (28) is sufficient (cf. [11], pp. 16-17). It is also seen from (33) and (28) that term-by-term integration of (35) on every closed half-line $x \geqq \epsilon$, where $\epsilon > 0$, is legitimate:

$$(36) \qquad \int_x^\infty f(u)du = \sum_{n=1}^\infty \mu(n) \int_x^\infty F(nu)du.$$

Hence, the assertion, (34), is equivalent to

$$(37) \qquad \lim_{x\to 0} \sum_{n=1}^\infty \mu(n) \int_x^\infty F(nu)du = 0.$$

But (37) follows from (26), (27) and (32) by the same partial summation which Hardy and Littlewood [5] apply in their proof of the implication $(L) \to (A)$, that is, of (α_0).

It should be emphasized that, due to the first part of (27), this proof is purely "Abelian" in nature. Correspondingly, (vii) is beyond the scope of the "Tauberian" technique of Karamata-Weiner, which would involve just the prime number theorem, that is, the second part of (27). In fact, this methodical situation prevails not only for (vii) but for (α_0) as well, and (α_0) is contained in (vii).

4. Before combining the facts preceding (vii) with the Fourier expansion, (5), of the kernel, $x - [x]$, of the Euler-Maclaurin integral, (11), it is convenient to replace (i) by the following variant of (ii):

(viii) *Let $f(x)$ be defined on the open half-line $x > 0$ in such a way that*

$$(38) \qquad f(x) \to 0 \text{ as } x \to \infty$$

and

$$(39) \qquad \epsilon f(\epsilon) \to 0 \text{ as } \epsilon \to 0,$$

and that the Lebesgue-Stieltjes integral of $u - [u]$ with respect to $f(u)$ exists on every closed, bounded interval, $\epsilon \leqq u \leqq x$. Suppose further that both improper integrals

$$(40) \qquad \int_{+0}^{1-0} u\,df(u) = \lim_{\epsilon\to 0} \int_\epsilon^{1-0}, \qquad \int_1^\infty (u - [u])df(u) = \lim_{x\to\infty} \int_1^x$$

are convergent (possibly just conditionally). Then the limit

$$(41) \qquad \lim_{x\to\infty}\Big(\sum_{n\leqq x} f(n) - \int_{+0}^{x} f(u)\,du\Big),$$

where $n=1,2,\cdots$, *exists and is equal to the value of*

$$(42) \qquad \int_{+0}^{\infty} (u-[u])\,df(u),$$

the sum of the two integrals (40).

In order to see this, it is sufficient to observe that $u-[u]=u$ when $0<u<1$, and that what therefore corresponds to (12) is

$$\int_{\epsilon}^{x} f(u)\,d([u]-u) = o(1)O(1) - o(1) - \int_{\epsilon}^{x} ([u]-u)\,df(u)$$

as $(\epsilon, x)\to(+0,\infty)$; cf. (38) and (39).

(ix) *Let* $f(x)$ *be of finite total variation on every closed half-line contained in the open half-line* $x>0$. *For* $x\to\infty$, *choose* $f(\infty)=0$. *For* $x\to+0$, *suppose that*

$$(43) \qquad \int_{\epsilon}^{\infty} |\,df(x)\,| = o(\epsilon^{-1}) \text{ as } \epsilon\to 0$$

and that the improper integral

$$(44) \qquad \int_{+0}^{1} f(x)\,dx \text{ is convergent}$$

(*possibly just conditionally*). *Then, if* $t>0$, *the limit*

$$(45) \qquad F^*(t) = \lim_{x\to\infty}\Big(t\sum_{nt\leqq x} f(nt) - \int_{+0}^{x} f(u)\,du\Big),$$

where $n=1,2,\cdots$, *exists and has the value of*

$$(46) \qquad F^*(t) = t\int_{+0}^{\infty} (x/t-[x/t])\,df(x),$$

and the improper integral

$$(47) \qquad f^*(t) = \int_{+0}^{\infty} f(x)\cos tx\,dx = \lim_{\epsilon\to 0}\int_{\epsilon}^{1} + \lim_{x\to\infty}\int_{1}^{x}$$

is convergent and represents a continuous function of $t>0$.

First, the assumptions of (ix) imply those of (viii). This is clear, except for the assumption (39) of (viii). But (39) follows from the remark made in connection with (23). On the other hand, if $t > 0$ is fixed, the assumptions of (ix) are satisfied for $f(tx)$ if they are satisfied for $f(x)$. Hence, the assumptions of (ix) imply that (viii) is applicable to $f(tx)$. This proves the existence of the limit (45), and the representation (46) of this limit.

Next, since $\sin(\epsilon t)$ is $O(\epsilon)$ when t is fixed, (38), (44) and (8) imply that the improper integral

$$\int_{+0}^{\infty} \sin tx \, df(x) \tag{48}$$

is convergent. On the other hand, if the integration range, $(+0, \infty)$, of (48) is replaced by the interval (ϵ, x), then, if $t \neq 0$, a partial integration, when followed by an application of (38) and (39), shows that the resulting approximation to (48) tends, as $(\epsilon, x) \to (+0, \infty)$, to a limit, and that the latter is $-t$ times the integral (47). Hence, in order to complete the proof of (ix), only the continuity of the function (47), where $t \neq 0$, remains to be ascertained. But this is clear from the fact that the preceding limit process is uniform on every closed, bounded t-interval not containing $t = 0$.

(x) *If $f(x)$ is a function of finite total variation on the half-line $x > 0$ and tends to 0 as $x \to \infty$* (that is, if the assumptions (43), (44) of (ix) are replaced by the stricter assumption

$$\int_{+0}^{\infty} |\, df(x)| < \infty \tag{49}$$

and $f(\infty) = 0$ is retained), *then the series*

$$\sum_{n=1}^{\infty} f^*(nt), \text{ where } f^*(t) = \int_0^{\infty} f(x) \cos tx \, dx, \tag{50}$$

is convergent for every $t > 0$ and is connected with the Euler-Maclaurin function (45) *by the relation*

$$F^*(t) = -\tfrac{1}{2} t f(+0) + 2 \sum_{n=1}^{\infty} f^*(2\pi n/t), \qquad (t > 0), \tag{51}$$

provided that $f(x)$ is normalized by

$$2f(x) = f(x+0) + f(x-0), \qquad (x > 0). \tag{52}$$

This normalization, made possible by (49), has no influence on the integrals occurring in (44) and (50).

The identity (51) is just Poisson's Fourier analysis (cf. [7], pp. 78-79) of the Euler-Maclaurin sum formula, first proved, and found independently, by Dirichlet. Under the above assumptions, it is an immediate consequence of (6) and (7).

First, if the value of (5) were always $\phi(x) = x - [x]$, then $\phi(x/t)$ could be substituted from (5) into (46). According to (6), the value of (5) ceases to be $\phi(x)$ when x is an integer. But this discrepancy has an effect on the Stieltjes integral (46) only at points at which f, too, is discontinuous. Hence, the discrepancy has no effect if f has no discontinuities at all. On the other hand, when f has jumps, then, since the latter are normalized by (52), the discrepancy is compensated by the fact that, if $\phi_0(x)$ denotes the value of (5), the normalization (52) holds, by (6), for $f = \phi_0$ also. Accordingly, the function, $\phi(x/t)$, which multiplies $df(x)$ in (46), can be replaced by $\phi_0(x/t)$.

Since (7) and (49) make it clear that term-by-term integration of the series $\phi_0(x/t)$ is legitimate in (46), it now follows from the expansion (5) that

$$F^*(t) = t\int_{+0}^{\infty} \tfrac{1}{2}df(x) - t\pi^{-1}\sum_{n=1}^{\infty} n^{-1}\int_{+0}^{\infty} \sin(2\pi nx/t)\,df(x).$$

But the first integral on the right is $-\frac{1}{2}f(+0)$, by (38); and, what concerns the second integral, the function (48) was seen to be identical with $-t$ times the function (47). This gives

$$F^*(t) = -\tfrac{1}{2}tf(+0) - t\pi^{-1}\sum_{n=1}^{\infty} n^{-1}f^*(2\pi n/t)(-2\pi n/t),$$

which is (51).

(xi) *If $f(x)$ is of finite total variation on the half-line $x > 0$ and if the improper integral*

(53) $$f^*(0) = \int_0^{\infty} f(x)\,dx = \lim_{x\to\infty}\int_0^{x}$$

is convergent, then

(54) $$\sum_{n=1}^{\infty} f^*(nt) \to 0 \text{ as } t \to \infty,$$

where $f^*(t) = \int_0^{\infty} f(x)\cos tx\,dx$, *whereas*

$$(55) \qquad t\sum_{n=1}^{\infty} f^*(nt) \to \tfrac{1}{2}\pi f(+0) \text{ as } t \to +0;$$

and, if $t > 0$,

$$(56) \qquad t^{\frac{1}{2}}\{\tfrac{1}{2}f(+0) + \sum_{n=1}^{\infty} f(nt)\} = t^{-\frac{1}{2}}\{f^*(0) + 2\sum_{n=1}^{\infty} f^*(2\pi n/t)\}.$$

The last identity is Cauchy's celebrated formula of reciprocity. It follows from (51) and (45), since the assumptions of (xi) imply those of (x), the convergence of (53) being assumed now. In fact, this assumption and (49) imply (38).

The proof of (54) can be based on the remark that (56) transforms (54) into

$$\epsilon\sum_{n=1}^{\infty} f(n\epsilon) \to f^*(0), \qquad \epsilon \to 0$$

(as seen if (56) is multiplied by $t^{\frac{1}{2}}$ and t is then identified by ϵ, finally the $2\pi/t$, occurring on the right of (55), is replaced by ϵ). In view of the definition, (53), of $f^*(0)$, the assertion of the last formula line is equivalent to (18). But (18) is applicable, since the assumptions of (xi) imply those of (iv). This proves (54).

The remaining assertion, (55), is a relation of Gibbs' type (in this regard, cf. [11], p. 5 and p. 28). It can be obtained by observing that, according to (49) and (53),

$$\sum_{n=1}^{\infty} f(nt) \to 0, \qquad t \to \infty.$$

In view of (56), this implies that

$$\tfrac{1}{2}f(+0) + o(1) = O(t^{-1}) + 2t^{-1}\sum_{n=1}^{\infty} f^*(2\pi n/t)$$

or, if t is replaced by $2\pi t$,

$$\tfrac{1}{2}f(+0) + o(1) = \pi^{-1}t^{-1}\sum_{n=1}^{\infty} f^*(n/t),$$

as $t \to \infty$. Since the last relation is equivalent to (55), the proof of (xi) is complete.

The limit relation (54) neither contains, nor is contained in, the estimate

$$(57) \qquad |f^*(t)| < \text{const.}/t \qquad (t > 0).$$

The latter holds even if the assumptions of (xi) are relaxed to those of (x). For, on the one hand, (48) is majorized by (49) and, on the other hand, (48) is $-t$ times the function (47).

The other limit relation, (55), can be interpreted as follows:

(xii) *If $f(x)$ satisfies the assumptions of* (xi), *then its Fourier cosine transform, $f^*(t)$, satisfies the equidistant Riemannian relation,*

$$\epsilon \sum_{n=1}^{\infty} f^*(n\epsilon) \to \int_{+0}^{\infty} f^*(t)\,dt \text{ as } \epsilon \to 0 \tag{58}$$

(the convergence of the improper integral

$$\int_{+0}^{\infty} f^*(t)\,dt = \lim_{\epsilon\to 0} \int_{\epsilon}^{1} + \lim_{x\to\infty} \int_{1}^{x}, \tag{59}$$

as well as the convergence of the series on the left of (58), *is part of the statement).*

In order to see this, let the function $f(x)$, given for $x > 0$, be extended for all x by placing $f(x) = 0$ when $x < 0$, and $f(0) = \frac{1}{2}f(+0)$. Then the assumptions of (xi) imply that

$$\int_{-\infty}^{\infty} |\, df(x)| < \infty \tag{60}$$

and

$$f(x) \to 0 \text{ as } x \to \pm\, \infty, \tag{61}$$

and that (52) holds for $-\infty < x < \infty$. But Pringsheim ([10], pp. 405-406; cf. also Hahn's presentation in [3], pp. 318-321) has shown that, if $f(x)$ satisfies (60), (61) and (52), then its representation by Fourier's "double integral" is valid for every x, provided that the exterior integration is interpreted as improper not only at ∞ but at 0 as well. If this is applied at $x = 0$ to the above extension of the given function $f(x)$, $0 < x < \infty$, then, since $f^*(t)$ is defined by (47), it follows that both parts of the improper integral (59) converge, and that the sum (59) has the value $\pi f(0)$.

The assumption that the integral (53) converges has not been used thus far. If it is used, (xi) becomes applicable, and so (58) follows from (55), where $\frac{1}{2}\pi f(+0) = \pi f(0)$.

Crucial in this deduction is the convergence of the improper integral (59) (along with the circumstance that (59) actually is improper at $t = 0$, in general), as is, correspondingly, that assumption of (xi) which is not required in (x), namely, the convergence of the improper integral (53). It is instructive to compare this situation with the proof of the following fact, which is more on the surface:

(xiii) *If $f(x)$ satisfies just the assumptions of* (x) (so that the additional restriction of (xi) and (xii), that requiring the convergence of (53), is not assumed), *then, as* $\epsilon \to 0$,

$$(62) \qquad \epsilon \sum_{n=1}^{\infty} f(n\epsilon) \cos(tn\epsilon) \to f^*(t), \text{ where } f^*(t) = \int_0^{\infty} f(x) \cos tx\, dx,$$

if $t \neq 0$ (and

$$(62\text{ bis}) \qquad \epsilon \sum_{n=1}^{\infty} f(n\epsilon) \sin t(n\epsilon) \to \int_0^{\infty} f(x) \sin tx\, dx,$$

where $t = 0$ need not, of course, be excluded).

In fact, (xiii) is a mere restatement of (vi), where $0 \neq \lambda = \pm t$. The case $0 = \lambda = t$ of (24), (62), excluded in (vi), (xiii), represents the more delicate assertion of (xii).

5. A sufficient criterion for the legitimacy of Möbius' inversion (*and* for the inversion of this inversion, which is not always the same thing; cf. [11], pp. 17-18) is as follows: If

$$(63) \qquad x_n = O(n^{-1-\eta})$$

is required for some $\eta > 0$, then the infinity of equations

$$(64) \qquad \sum_{m=1}^{\infty} x_{nm} = y_n$$

for the infinity of unknowns $x_1, x_2, \cdots$ has the unique solution

$$(65) \qquad x_n = \sum_{m=1}^{\infty} \mu(m) y_{nm},$$

(no matter what the sequence of data $y_1, y_2, \cdots$ is), where μ denotes Möbius' factor; whereas, if (63) is omitted and

$$(66) \qquad y_n = O(n^{-1-\delta})$$

is required for some $\delta > 0$, then (65), when considered as a system for the unknowns $y_1, y_2, \cdots$, has a unique solution, which is precisely the y_n supplied by (64) (no matter what the sequence of data $x_1, x_2, \cdots$, given in (65), is this time).

Somewhat less (but not very much less; cf. [11], pp. 16-18) than the assumption of an $\eta > 0$ or a $\delta > 0$ is sufficient. The proof requires, of course, nothing but formal rearrangements which, under the respective restrictions

(63), (64), are justified by the absolute convergence of the repeated summations involved.

It should be noted that, by virtue of (64) and (65), *the existence of an* $\eta > 0$ *is equivalent to the existence of a* $\delta > 0$, and that, as a matter of fact,

$$\text{l. u. b. } \eta = \text{l. u. b. } \delta, \tag{67}$$

if either of the Dedekind cuts occurring in (67) is positive (possibly ∞). This is seen by substituting (63) into (64), and (66) into (65), where $\mu(m) = O(1)$.

The symmetry relation (67) will be used below to an end corresponding to the one-to-one correspondences, developed in [11], pp. 25-26, between the smoothness of a periodic R-integrable function and the rapidity of the convergence of its equidistant R-sums.

For the sake of convenient references, let the preceding remarks be summarized in a slightly different form, as follows:

(*) *If* t *varies on the half-line* $t > 0$, *and if either*

$$g(t) = O(t^{-1-\eta}), \qquad t \to \infty, \tag{68}$$

holds for some $\eta > 0$ *or*

$$h(t) = O(t^{-1-\delta}), \qquad t \to \infty, \tag{69}$$

holds for some $\delta > 0$, *then*

$$h(t) = \sum_{n=1}^{\infty} g(nt), \qquad 0 < t < \infty, \tag{70}$$

is equivalent to

$$g(t) = \sum_{n=1}^{\infty} \mu(n) h(nt), \qquad 0 < t < \infty, \tag{71}$$

and, by virtue of the reciprocal relations (69), (70), *either of the restrictions* (68), (69) *implies the other restriction; in fact, even the "best" values of the exponents* η, δ *are identical (in the sense of* (67), *where* l. u. b. > 0 *is assumed and* l. u. b. $= \infty$ *is allowed).*

In order to verify this criterion, (*), it is sufficient to replace t in (70) and (71) by kt, where $k = 1, 2, \cdots$; to identify the resulting systems with (60), (65) (for a fixed t), and (68), (69) with (63), (66), respectively;

finally, to particularize the parametric integer ($=k$) to be 1 in the resulting inversions.

The *self-reciprocal* relationship expressed by (69), (68) and (67) has variants on other scales; for instance, on the following exponential scale:

(* bis) *The assertions of* (*) *remain true if* (67) *is retained but* (68) *and* (69) *are replaced by*

$$g(t) = O(\exp - t^{\eta}), \qquad t \to \infty \tag{70 bis}$$

and

$$h(t) = O(\exp - t^{\delta}), \qquad t \to \infty, \tag{71 bis}$$

respectively, where $\eta > 0$ *and* $\delta > 0$.

In fact, since the assumptions of (* bis) imply those of (*), it is sufficient to ascertain that (67) holds in the case of (* bis) also. But this follows in the same way as before; namely, by substituting the O-assumptions into the respective series (70), (71), where $\mu(n) = O(1)$, as $n \to \infty$, and then using the fact that, as $t \to \infty$, both estimates

$$\text{(72)} \quad \sum_{n=1}^{\infty} (nt)^{-1-\epsilon} = O(t^{-1-\epsilon}), \qquad \text{(73)} \quad \sum_{n=1}^{\infty} \exp - (nt)^{\epsilon} = O(\exp - t^{\epsilon}),$$

the first of which belongs to (*) and the second to (* bis), hold for every fixed $\epsilon > 0$.

These Möbius criteria will now be combined with the consequences of the Euler-Maclaurin lemma (ix) and its Fourier corollaries, deduced above from (5), (6), (7). The simplest fact which thus results can be isolated as follows:

(xiv) *If* $f(x)$, *where* $x > 0$, *satisfies the conditions*

$$\int_0^{\infty} | df(x) | < \infty \text{ and } f(\infty) = 0, \tag{74}$$

and if $f^*(t)$, $F^*(t)$ *denote the functions which* (47), (45) *then define for* $t > 0$, *the existence of an* α *satisfying*

$$\alpha > 1 \text{ and } f^*(t) = O(t^{-\alpha}) \text{ as } t \to \infty \tag{75}$$

is necessary and sufficient for the existence of a β *satisfying*

(76) $\beta > 1$ and $F^*(x) = -\frac{1}{2}f(+0)x + O(x^\beta)$ as $x \to +0$;

in fact, (74) *and either of the assumptions* (75), (76) *imply that*

(77) $$\alpha^* = \beta^*,$$

where α^*, β^* *denote the least upper bounds* ($\leqq \infty$) *of the admissible values of the respective indices* α, β.

This is quite a refined manifestation of a general principle formulated by Paul Lévy [8], pp. 264-266.

The proof will be based on (*) in a manner which will make it clear that the replacement of (*) by (* bis) leads to the following variant of (xiv):

(xiv bis) *The assertions of* (xiv) *remain true if* (75) *is replaced by*

(75 bis) $\alpha > 0$ and $f^*(t) = O(\exp - t^\alpha)$ as $t \to \infty$,

and (76) *by*

(76 bis) $\beta > 0$ and $F^*(x) = -\frac{1}{2}f(+0)x + O(\exp - x^{-\beta})$ as $x \to +0$;

(it being understood that α^* and β^* in (77) now refer to (75 bis) and (76 bis), respectively).

First, the assumptions of (xiv) are those of (x), if $f(x)$ is normalized by (52). Hence, (51) is applicable. Let (51) be written in the form

(78) $$h(t) = \sum_{n=1}^{\infty} f^*(2\pi nt),$$

where $h(t)$ is defined by

(79) $$2h(t) = F^*(1/t) + \tfrac{1}{2}f(+0)/t, \qquad (t > 0).$$

Next, suppose that there exists an α satisfying (75). Then (72) shows that (69) is fulfilled by the function (78). Hence, if (78) is identified with (70), where $g(t) = f^*(2\pi t)$, it follows from (*) that both (71) and (68) hold in the present case. But (71) now appears in the form

$$f^*(2\pi t) = \sum_{n=1}^{\infty} \mu(n) h(nt),$$

whereas (79) shows that (68) is identical with (76).

On the other hand, if (75) is replaced by (76) as an assumption, then (79) shows that (76) is satisfied. Hence, if (78) is identified, as before, with (70), it is seen from (79) that (68) is fulfilled by $g(t) = f^*(2\pi t)$, which means that (75) is satisfied.

This proves (xiv), since the remaining assertion, (77), is clear from (67). In addition, the last two formula lines supply the following by-product:

(xv) *If* (52), (74) *and either* (75) *or* (76) *are satisfied by* $f(x)$, *then the Möbius expansion,*

$$(80) \qquad f^*(2\pi/t) = \tfrac{1}{2}\sum_{n=1}^{\infty}\mu(n)F^*(t/n),$$

of the Fourier cosine transform, (47), *in terms of the Euler-Maclaurin function,* (45), *is valid for every* $t > 0$.

Actually, the by-product is

$$(80') \qquad f^*(2\pi/t) = \tfrac{1}{2}\sum_{n=1}^{\infty}\mu(n)\{F^*(t/n) + \tfrac{1}{2}f(+0)t/n\},$$

and so (80) follows only if either $f(+0)$ is 0 or recourse is had to

$$(81) \qquad \sum_{n=1}^{\infty}\mu(n)/n = 0,$$

that is, to the prime number theorem. This means that (xv) *is elementary or just as deep as the prime number theorem according as* $f(+0) = 0$ *or* $f(+0) \neq 0$.

In (xv), the assumptions are the same as in (x). Under the additional assumption made in (xi), the assertion of (xv) can be dualized, as follows:

(xvi) *If* (52), (74) *and* (28) *are satisfied by* $f(x)$, *and if the Fourier cosine transform,*

$$(82) \qquad f^*(t) = \int_0^{\infty} f(x)\cos tx\,dx = \lim_{x\to\infty}\int_0^{x},$$

remains convergent at $t = 0$, *then the Euler-Maclaurin function,*

$$(83) \qquad F^*(t) = t\sum_{n=1}^{\infty}f(nt) - \int_0^{\infty} f(x)\,dx,$$

provides a Möbius expansion,

(84) $$xf(x) = \sum_{n=1}^{\infty} \mu(n) F^*(nx)/n,$$
of $f(x)$ for every $x > 0$.

What will now follow directly, namely,

(84′) $$xf(x) = \sum_{n=1}^{\infty} \mu(n) \{F^*(nx) + f^*(0)\}/n,$$

will reduce to (84) *without* recourse to (81) only if $f^*(0) = 0$; so that (xvi) *is elementary or involves precisely the prime number theorem according as the value of the integral* (53) *is or is not* 0.

The assumptions of (xvi) are identical with those of (xi). Hence, (56) is applicable. On the other hand, (53) shows that (83) can be written in the form

(78′) $$h(t) = \sum_{n=1}^{\infty} f(nt),$$
if $h(t)$ is defined by

(79′) $$h(t) = F^*(t)/t + f^*(0)/t.$$

The last two formula lines take over the parts played by (78) and (79) in the proof of (xiv), (xv). In fact, it is clear that the balance of the proof of (xvi) is the same as, via (*) and (xiv), the proof of (xv) was.

6. An instance of explicit interest will now be considered: $f(x) = e^{-x^\lambda}$, where $\lambda > 0$. Then, since

$$f^*(0) = \int_0^\infty f(x)dx = \int_0^\infty e^{-x^\lambda}dx = \Gamma(1 + 1/\lambda),$$

(83) becomes $F^*(t) = t \sum_{n=1}^{\infty} e^{-t^\lambda n^\lambda} - \Gamma(1 + 1/\lambda)$. This means that $F^*(t^{1/\lambda})/t^{1/\lambda}$ is the function

$$\sum_{n=1}^{\infty} e^{-tn^\lambda} - \Gamma(1 + 1/\lambda).$$

But the latter function is known to be the entire function

$$\sum_{m=0}^{\infty} (-1)^m \zeta(-\lambda m) t^m/m!,$$

if $0 < \lambda < 1$; cf. Mellin [9], p. 12. Hence, (80) becomes

$$f^*(2\pi/t) = \tfrac{1}{2}\sum_{n=1}^{\infty}\mu(n)\sum_{m=0}^{\infty}(-1)^m\zeta(-\lambda m)(t/n)^{\lambda m+1}/m!,$$

where $0 < \lambda < 1$.

The contribution of $m = 0$ to this repeated sum is

$$\tfrac{1}{2}\sum_{n=1}^{\infty}\mu(n)\zeta(0)t/n = \tfrac{1}{2}t\zeta(0)\sum_{n=1}^{\infty}\mu(n)/n = 0,$$

by (81). Hence,

$$f^*(2\pi/t) = \tfrac{1}{2}t\sum_{n=1}^{\infty}\mu(n)\sum_{m=0}^{\infty}(-t^\lambda)^m \ \zeta(-\lambda m)/(m!n^{\lambda m+1}).$$

This result is independent of the prime number theorem. For, on the one hand, (81) has been used twice (first in the reduction of (80′) to (80) and then in the omission of the summation value $m = 0$) and, on the other hand, it is clear that these two steps can be united into one which avoids (81) entirely.

Since $\mu(n) = O(1)$, it is easily seen (from the behavior of $\zeta(-x)$ for large positive x) that the repeated sum on the right of the last formula line actually is an absolutely convergent double series; in fact, as mentioned before, Mellin's power series converges for arbitrarily large t, since $0 < \lambda < 1$. Hence, the order of summations can be interchanged. The contribution of what then becomes the interior summation is

$$\sum_{n=1}^{\infty}\mu(n)(-t^\lambda)^m \ \zeta(-\lambda m)/(m!n^{\lambda m+1}) = (-t^\lambda)^m/m! \ \zeta(-\lambda m)/\zeta(\lambda m+1),$$

since $\sum_{n=1}^{\infty}\mu(n)/n^s = \zeta(s)$ when $s > 1$. Accordingly,

$$f^*(2\pi/t) = \tfrac{1}{2}t\sum_{n=1}^{\infty}(-t^\lambda)^n/n! \ \zeta(-\lambda m)/\zeta(\lambda m+1).$$

But $f(x)$ was the function e^{-x^λ}; so that $f^*(t) = \int_0^\infty e^{-x^\lambda}\cos tx\,dx$, by (82). Consequently,

$$\int_0^\infty e^{-x^\lambda}\cos 2\pi tx\,dx = \tfrac{1}{2}t^{-1}\sum_{n=1}^{\infty}\zeta(-\lambda n)/\zeta(\lambda m+1)(-t^{-\lambda})^n/n!.$$

The coefficients of this power series can be reduced by using Riemann's functional equation,

$$\tfrac{1}{2}\zeta(1-s) = (2\pi)^{-s}\zeta(s)\Gamma(s)\cos\tfrac{1}{2}\pi s.$$

For $s = 1 + \lambda n$, this gives

$$\tfrac{1}{2}\zeta(-\lambda n)/\zeta(1+\lambda n) = (2\pi)^{-1}(2\pi)^{-\lambda n}\Gamma(1+\lambda n)(-\sin\tfrac{1}{2}\pi\lambda n).$$

Hence, the preceding representation of the Fourier cosine integral $f^*(2\pi t)$ can be written in the form

$$-(2\pi t)^{-1}\sum_{n=1}^{\infty}\Gamma(1+\lambda n)\sin\tfrac{1}{2}\pi\lambda n\ (-t^{-\lambda})^n(2\pi)^{-\lambda n}/n!.$$

Accordingly, the final result is as follows:

If $0<\lambda<1$, then

$$\int_0^\infty e^{-x^\lambda}\cos tx\,dx = \sum_{n=1}^{\infty}(-1)^{n+1}\sin\tfrac{1}{2}\pi\lambda n\Gamma(\lambda n+1)/n!\ t^{-\lambda n-1},$$

where t is arbitrary ($\neq 0$). Cf. [12].

In the limiting case $\lambda = 1$, this expansion is equivalent to that of $(1+t^{-2})^{-1}$, a geometric progression valid for certain, but not for all, values of t. It is easy to see from the above proof that, if $\lambda > 1$, the power series, which then diverges everywhere, is an asymptotic expansion (as $t\to\infty$) of the function $f^*(t)$. It should be noted that, if λ is an even integer, then all coefficients vanish, and so

$$f^*(t) = O(t^{-N}) \text{ as } t\to\infty,$$

where N is arbitrarily large.

A final remark is needed for the justification of the above application of (80), since (xv) assumes either (75) or (76). But (75), with $\alpha = 1+\lambda$, is clear from the first term of the final expansion, which can, of course, be verified directly, and also (76) can be verified directly, by using the indications of Lévy, referred to after (xiv); in fact, Lévy formulates his general principle precisely in connection with the above $f^*(t)$, the Fourier transform of a symmetric stable distribution (cf. [8], pp. 264-277).

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APPLICATIONS OF INDUCED CHARACTERS.*

By Richard Brauer.

1. Introduction. In a previous paper,[1] the following theorem on induced characters of groups was proved: *If $\mathfrak{G}$ is a group of finite order g, every character χ of $\mathfrak{G}$ can be written in the form $\chi = \Sigma a_\rho \omega_\rho^*$ where the a_ρ are rational integers and where the ω_ρ^* are characters of $\mathfrak{G}$ induced by linear characters ω_ρ of subgroups $\mathfrak{H}_\rho$ of $\mathfrak{G}$.* If we call a group *elementary,* if it is a direct product $\mathfrak{U} \times \mathfrak{B}$ of a group $\mathfrak{U}$ of prime power order and a cyclic group $\mathfrak{B}$ of an order prime to the order of $\mathfrak{U}$, then we may assume that all the groups $\mathfrak{H}_\rho$ are elementary groups. This can be seen at once from the proof of the theorem.

As an immediate consequence of this result, it will be shown in **2** that every representation of a group $\mathfrak{G}$ of order g can be written in the field of the g-th roots of unity. Our new approach to this problem is simpler and more elementary than that given in an earlier investigation.[2] At the same time it yields stronger results. For instance, if n is the least common multiple of the orders of the elements of $\mathfrak{G}$, then every representation of $\mathfrak{G}$ can be written in the field of the n-th roots of unity.

3 deals with a method of determining the irreducible characters of a group of finite order. If ω is a character of a subgroup $\mathfrak{H}$ of $\mathfrak{G}$, we need the following information in order to be able to construct the character ω^* of $\mathfrak{G}$. We have to know (a) the number k of classes $\mathfrak{K}_1, \mathfrak{K}_2, \cdots, \mathfrak{K}_k$ of conjugate elements in $\mathfrak{G}$ and the number g_i of elements in $\mathfrak{K}_i$; (b) the number l of classes $\mathfrak{L}_1, \mathfrak{L}_2, \cdots, \mathfrak{L}_l$ of conjugate elements in $\mathfrak{H}$ and the number h_j of elements in $\mathfrak{L}_j$; (c) the value $i = i(j)$ such that $\mathfrak{L}_j \subseteq \mathfrak{K}_{i(j)}$, $j = 1, 2, \cdots, l$; (d) the value $\omega(\mathfrak{L}_j)$ of ω for $\mathfrak{L}_j$.[3] If this information is given for all elementary subgroups $\mathfrak{H}$ of $\mathfrak{G}$ and all linear characters ω of $\mathfrak{H}$, then it is shown that all irreducible characters of $\mathfrak{G}$ can be constructed. In order to obtain all linear characters ω of $\mathfrak{H}$, we have to know the normal subgroups $\mathfrak{H}_0$ of $\mathfrak{H}$ with

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[1] R. Brauer, *Annals of Mathematics*, vol. 48 (1947), pp. 502-514.

[2] R. Brauer, *American Journal of Mathematics*, vol. 67 (1945), pp. 461-471.

[3] If ω is a character of $\mathfrak{H}$, H an element of the class $\mathfrak{L}$ of $\mathfrak{H}$, we denote by $\omega(\mathfrak{L})$ the value $\omega(H)$ taken by ω for the element H.

cyclic factor group $\mathfrak{H}/\mathfrak{H}_0 = \{\mathfrak{H}_0 F\}$ and we have to know the exponent $\rho = \rho(j)$ for which the class $\mathfrak{L}_j$ belongs to the coset $\mathfrak{H}_0 F^\rho$, $j = 1, 2, \cdots, l$. It seems remarkable that it is possible to construct the characters of $\mathfrak{G}$ on the basis of so little information concerning the structure of $\mathfrak{G}$.

In **4**, an analogue of the theorem on induced characters for the case of modular characters is given. If p^a is the highest power of the prime p dividing the group order g, then as an application the number N of ordinary irreducible representations $\mathfrak{Z}$ of $\mathfrak{G}$ is determined whose degree z is divisible by p^a. The number N is obtained as the number of representations of 1 by means of a quadratic form with integral rational coefficients.

2. Representation of groups in cyclotomic fields. We first prove:

THEOREM 1. *Let $\mathfrak{G}$ be a group of finite order and let n be the least common multiple of the orders of the elements of $\mathfrak{G}$. Every representation of $\mathfrak{G}$ can be written in the field of the n-th roots of unity.*

Proof. It is sufficient to prove the theorem for irreducible representations $\mathfrak{Z}$ of $\mathfrak{G}$. The field K of the n-th roots of unity certainly contains the character χ of $\mathfrak{Z}$ as well as all characters ω of subgroups $\mathfrak{H}$ of $\mathfrak{G}$. In particular, every linear representation $\mathfrak{M}$ of a subgroup $\mathfrak{H}$ lies in K, and the same holds for the representation $\mathfrak{M}^*$ of $\mathfrak{G}$ induced by $\mathfrak{M}$. According to the theorem on induced characters quoted in the introduction, we have

$$\chi = \Sigma a_\rho \omega_\rho^* \tag{1}$$

where the a_ρ are rational integers and where the ω_ρ^* are characters induced by linear characters ω_ρ of subgroups $\mathfrak{H}_\rho$ of $\mathfrak{G}$. If p is a fixed prime number, then (1) shows that there exists at least one ω_ρ^* which contains χ with a multiplicity t prime to p. The representation $\mathfrak{M}^*$ belonging to ω_ρ^* lies in the field K and contains $\mathfrak{Z}$ with the multiplicity t. This implies [4] that the Schur index m of $\mathfrak{Z}$ with respect to K divides t and is, therefore, prime to p. Since this holds for every prime p, we have $m = 1$. Then, $\mathfrak{Z}$ can be written in the field K as was to be shown.

The same procedure yields a slightly better result. Let K_0 be the field $\mathrm{P}(\chi)$ obtained by adjunction of the character χ of $\mathfrak{Z}$ to the field P of rational numbers, and let m_0 be the Schur index of $\mathfrak{Z}$ with respect to K_0. As was shown by Schur,[5] m_0 divides the degree z of $\mathfrak{Z}$. If p is a prime factor of m_0, we wish to adjoin an α-th root of unity to K_0 such that the Schur index of $\mathfrak{Z}$

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[4] See I. Schur, Sitzungberichte Preuss. Akad. Wiss. (1906), pp. 164-184.

[5] *Loc. cit.*[4]

with respect to the extended field is no longer divisible by p. Choose as above $\omega_\rho{}^*$ in (1) such that $\omega_\rho{}^*$ contains χ with a multiplicity t not divisible by p. If the values of ω_ρ lie in the field of the α-th roots of unity, then the adjunction of the α-th roots of unity to K_0 will have the desired effect. However, this adjunction is equivalent to the simultaneous adjunction of roots of unity of prime power exponents q^b where all q^b divide α. If $q \not\equiv 0, 1 \pmod p$, the degree of a (q^b)-th root of unity with respect to K_0 is not divisible by p. Hence the adjunction of the (q^b)-th roots of unity cannot change the power of p in the Schur index in this case. Similarly, if $q \neq p$, $b > 1$, the adjunction of the (q^b)-th roots of unity can be replaced by adjunction of the q-th roots of unity and the same reduction of the power of p in the Schur index will be achieved. Finally, if $q = p$, $b = 1$, the adjunction of the corresponding roots of unity is again superfluous. We thus see that we may replace α by a divisor β which contains only prime factors q of the form $q \equiv 0, 1 \pmod p$, the factors $q \equiv 1$ all with the exponent 1, and the factor $q = p$ with an exponent $b \neq 1$ (possibly $b = 0$). Since we may assume that $\mathfrak{G}$ contains elements of the order α, it will also contain elements of order β.

If this procedure is applied for all prime divisors of m_0, the following theorem is obtained.

THEOREM 2. *Let $p_1, p_2, \cdots, p_r$ be the distinct primes which divide the Schur index of the irreducible representation $\mathfrak{Z}$ of $\mathfrak{G}$ with respect to the field of the character of $\mathfrak{Z}$. (Then the p_ρ divide the degree of $\mathfrak{Z}$.) We can find a system of elements $G_1, G_2, \cdots, G_r$ of $\mathfrak{G}$ with the following properties:*

1. *The order β_ρ of G_ρ contains only primes $q \equiv 0, 1 \pmod{p_\rho}$. If the prime p_ρ appears in β_ρ, it appears with an exponent $\geqq 2$. All other prime factors of β_ρ appear only with the exponent* 1.

2. *If v is the least common multiple of $\beta_1, \beta_2, \cdots, \beta_r$ then $\mathfrak{Z}$ can be written in the field which is obtained from the field of the character of $\mathfrak{Z}$ by an adjunction of a v-th root of unity.*

We add a remark for the case that the character χ of the representation $\mathfrak{Z}$ is real. If the degree z of $\mathfrak{Z}$ is odd, then A. Speiser[6] showed that $\mathfrak{Z}$ can be written in the field K_0 of the character χ. If z is even, the Schur index m_0 divides 2.[7] It is then sufficient to consider only the prime 2 in Theorem 2.

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[6] A. Speiser, *Mathematische Zeitschrift*, vol. 5 (1919), pp. 1-6.

[7] R. Brauer, Sitzungberichte Preuss. Akad. Wiss. (1926), pp. 410-416; R. Brauer, H. Hasse, and E. Noether, *Journal für die reine und angewandte Mathematik*, vol 167

As is easily seen, the number β_1 can be taken here either as an odd prime or as a power of 2. This gives

THEOREM 3. *If $\mathfrak{Z}$ is an irreducible representation with a real character, there exists an element G in $\mathfrak{G}$ whose order β is either an odd prime or a power of 2, such that $\mathfrak{Z}$ can be written in the field obtained by adjunction of the β-th roots of unity to the field of characters.*

3. Construction of the characters of a group $\mathfrak{G}$ of finite order. If $\mathfrak{H}$ is a subgroup of order h of $\mathfrak{G}$ and if ω is a character of $\mathfrak{H}$, the induced character ω^* of $\mathfrak{G}$ is given by

$$\omega^*(G) = (1/h)\Sigma\omega(RGR^{-1}) \tag{2}$$

where R on the right ranges over all g elements of $\mathfrak{G}$, and where $\omega(X) = 0$ if X is not an element of $\mathfrak{H}$. If G belongs to the class $\mathfrak{K}_i$ of $\mathfrak{G}$, only terms $\omega(\mathfrak{L}_j)$ will appear on the right side of (2) for which $\mathfrak{L}_j$ is a class of $\mathfrak{H}$ which is contained in $\mathfrak{K}_i$. If $\mathfrak{K}_i$ contains g_i elements and $\mathfrak{L}_j$ contains h_j elements, the term $\omega(\mathfrak{L}_j)$ appears with the multiplicity gh_j/g_i and (2) may be written in the form

$$\omega^*(\mathfrak{K}_i) = (g/hg_i)\Sigma h_j\omega(\mathfrak{L}_j), \tag{3}$$

the sum extending over all classes $\mathfrak{L}_j$ of $\mathfrak{H}$ with $\mathfrak{L}_j \subseteq \mathfrak{K}_i$.

It is now evident that if the information (a), (b), (c), (d) mentioned in the introduction is given, the character ω^* as a function of $\mathfrak{K}_i$ is completely determined. Apply this for all elementary subgroups $\mathfrak{H}$ of $\mathfrak{G}$ and for all linear characters ω of $\mathfrak{H}$. In this manner, we obtain a system of characters $\omega_1^*, \omega_2^*, \cdots, \omega_r^*$ of $\mathfrak{G}$ such that every character of $\mathfrak{G}$ can be written in the form $\chi = \Sigma a_\rho\omega_\rho^*$ with integral rational coefficients.

Conversely, every ω_ρ^* can be written as a linear combination of the irreducible characters $\chi_1, \chi_2, \cdots, \chi_k$ of $\mathfrak{G}$ with integral rational coefficients. The same holds for a linear combination $\xi = \Sigma x_\rho\omega_\rho^*$ with integral rational coefficients x_ρ, say

$$\xi = \sum_{\rho=1}^{r} x_\rho\omega_\rho^* = \sum_{\kappa=1}^{k} u_\kappa\chi_\kappa. \tag{4}$$

The orthogonality relations for group characters yield

$$(1/g)\sum_i g_i\xi(\mathfrak{K}_i)\bar{\xi}(\mathfrak{K}_i) = (1/g)\sum_{\rho,\sigma} x_\rho x_\sigma \sum_i g_i\omega_\rho^*(\mathfrak{K}_i)\bar{\omega}_\sigma^*(\mathfrak{K}_i) = \sum_k u_\kappa^2.$$

Set

(1932), pp. 399-404. In the first of these papers, it was shown that the exponent of the representation is 2, and in the second paper that the exponent is equal to the Schur index.

(5) $$m_{\rho\sigma} = (1/g) \sum g_\iota \omega_\rho^*(\mathfrak{K}_\iota)\bar{\omega}_\sigma(\mathfrak{K}_\iota).$$

Then $m_{\rho\sigma}$ is a non-negative rational integer and we have

(6) $$\sum_{\rho,\sigma} x_\rho x_\sigma m_{\rho\sigma} = \sum_\kappa u_\kappa^2.$$

In particular, if the expression (6) is equal to 1, then either ξ or $-\xi$ is an irreducible character χ_κ of $\mathfrak{G}$. It is easy to decide which of these two cases we have. Indeed, let $\mathfrak{K}_1$ be the class which contains the 1-element of $\mathfrak{G}$. If $\xi(\mathfrak{K}_1) > 0$, then ξ itself is an irreducible character while in the other case $-\xi$ is an irreducible character.[8]

In order to find all irreducible characters of $\mathfrak{G}$, we have to find all solutions of the Diophantine equation

(7) $$\sum x_\rho x_\sigma m_{\rho\sigma} = 1$$

in rational integers x_ρ. The coefficients $m_{\rho\sigma}$ of the quadratic form on the left side can be found, if the ω_ρ^* are known. Only solutions x_ρ are to be used for which $\Sigma x_\rho \omega_\rho^*(\mathfrak{K}_1) > 0$. There are exactly k distinct expressions $\xi = \Sigma x_\rho \omega_\rho^*$ formed by means of such solutions x_ρ. These k expressions are the k irreducible characters of $\mathfrak{G}$.

THEOREM 4. *Suppose that the number k of classes $\mathfrak{K}_1, \mathfrak{K}_2, \cdots, \mathfrak{K}_k$ of conjugate elements of $\mathfrak{G}$ and the number g_ι of elements in $\mathfrak{K}_\iota$ are known. Suppose that a complete system of elementary subgroups $\mathfrak{H}$ of $\mathfrak{G}$ is given, (subgroups conjugate in $\mathfrak{G}$ may be considered as not essentially different). Assume further that for each $\mathfrak{H}$ the number l of classes $\mathfrak{L}_1, \mathfrak{L}_2, \cdots, \mathfrak{L}_l$ of conjugate elements of $\mathfrak{H}$ and the number h_j of elements of $\mathfrak{L}_j$ is known, that it is known to which class $\mathfrak{K}_\iota$ the elements of $\mathfrak{L}_j$ belong and that the values $\omega(\mathfrak{L}_j)$ of the linear characters ω of $\mathfrak{H}$ are known. Then the irreducible characters of $\mathfrak{G}$ are completely determined.*

As already remarked in the introduction, the construction of all linear characters $\omega(\mathfrak{L}_j)$ of $\mathfrak{H}$ requires only the knowledge of all normal subgroups $\mathfrak{H}_0$ with cyclic factor group of the elementary group $\mathfrak{H}$ and the information to which particular coset (mod $\mathfrak{H}_0$) the class $\mathfrak{L}_j$ belongs.

Since the characters $\omega_1^*, \omega_2^*, \cdots, \omega_r^*$ are, in general, linearly dependent, the equation (7) has, in general, infinitely many solutions. However, it can be seen without difficulty that the solutions can be found in a finite number of steps.

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[8] This method has been used by I. Schur in order to find the characters of special groups.

4. An analogue for modular characters. Let p now be a fixed prime number. We prove

THEOREM 5. *If Φ is the character of an indecomposable constituent of the modular regular representation of $\mathfrak{G}$ (mod p), then Φ can be written in the form*

$$\Phi = \sum_{\sigma} a_\sigma \omega_\sigma{}^*$$

where the a_σ are rational integers and where the $\omega_\sigma{}^*$ are characters of $\mathfrak{G}$ induced by linear characters ω_σ of elementary subgroups $\mathfrak{H}_\sigma$ of orders prime to p.

Proof. We first observe that Φ may be considered as an ordinary (reducible or irreducible) character of $\mathfrak{G}$ which vanishes for the p-singular classes of $\mathfrak{G}$.[9] All characters $\omega_\sigma{}^*$ in Theorem 5 vanish for the same classes.

As in the proof of the theorem on induced characters (see [1]), it is sufficient to show that if a congruence

$$\sum c_i \omega_\sigma{}^*(\mathfrak{K}_i) \equiv 0 \pmod{\mathfrak{q}^r} \tag{8}$$

modulo a prime ideal power $\mathfrak{q}^r$ of a suitable algebraic number field has $\mathfrak{q}$-integral coefficients c_i and holds for all characters $\omega_\sigma{}^*$ in Theorem 5, then the corresponding congruence

$$\sum c_i \Phi(\mathfrak{K}_i) \equiv 0 \pmod{\mathfrak{q}^t} \tag{9}$$

holds for Φ. It is sufficient to restrict the summation to p-regular classes $\mathfrak{K}_i$.

If $\mathfrak{q}$ does not divide p, it follows at once from Theorem 2 of the paper quoted in [1] that (8) implies (9). It remains to treat the case that $\mathfrak{q}$ is a prime ideal divisor of p. Let A be an element of $\mathfrak{K}_i$ and let $\xi_0, \xi_1, \cdots, \xi_{\alpha-1}$ denote the linear characters of the cyclic group $\{A\}$. Set

$$\psi(A^\nu) = \sum_{i=0}^{\alpha-1} \bar{\xi}_i(A) \xi_i(A^\nu).$$

Then

$$\psi(A^\nu) = \begin{cases} \alpha, & A^\nu = A \\ 0, & A^\nu \neq A \end{cases}. \tag{10}$$

The induced expression is $\psi^*(G) = (1/\alpha)\Sigma\psi(RGR^{-1})$ where R ranges over all elements of $\mathfrak{G}$. Now, (10) yields

$$\psi^*(G) = \begin{cases} n(A), & G \text{ in } \mathfrak{K}_i \\ 0\ , & G \text{ not in } \mathfrak{K}_i \end{cases}.$$

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[9] Cf. R. Brauer and C. Nesbitt, *Annals of Mathematics*, vol. 42 (1942), pp. 556-590, in particular, equation (9) and the argument in § 14.

where $n(A) = g/g_i$ is the order of the normalizer of A in $\mathfrak{G}$. Substituting this for ω_σ^* in (8), we find

$$c_i n(A) \equiv 0 \pmod{\mathfrak{q}^t}.$$

Now $\Phi(\mathfrak{K}_i)$ is divisible by the highest power of $\mathfrak{q}$ dividing $n(A)$. Hence

$$c_i \Phi(\mathfrak{K}_i) \equiv 0 \pmod{\mathfrak{q}^t}.$$

This implies (9), and Theorem 5 is proved.

The linear combinations of the characters $\omega_1^*, \omega_2^*, \cdots, \omega_s^*$ with integral rational coefficients form a module Ω. If elements of Ω are linearly dependent, there exists a linear relation with integral rational coefficients. This is seen at once when the elements of Ω are expressed by the irreducible characters of $\mathfrak{G}$. Let $\psi_1, \psi_2, \cdots, \psi_w$ be a basis of Ω. Then $\psi_1, \psi_2, \cdots, \psi_w$ are linearly independent in the field of all numbers.

In particular, the characters $\Phi_1, \Phi_2, \cdots$ of the distinct indecomposable constituents of the modular regular representation of $\mathfrak{G}$ belong to Ω and they are linearly independent. Every element of Ω vanishes for all p-singular classes $\mathfrak{K}_i$ of $\mathfrak{G}$ and can, therefore, be expressed by the Φ_i with integral rational coefficients.[10] Hence the Φ_i also form a basis of Ω. This shows that the number w of basis elements of Ω is equal to the number of distinct Φ_i, that is, to the number of p-regular classes $\mathfrak{K}_i$ in $\mathfrak{G}$. Further, the ψ_i and the Φ_i are connected by a unimodular linear transformation with integral rational coefficients,

$$\psi_i = \sum b_{ij}\Phi_j. \tag{11}$$

While it is of course possible to determine a basis $\psi_1, \psi_2, \cdots, \psi_w$ of Ω when the ω_σ^* are known, it seems that the Φ_j themselves cannot always be determined on the basis of this information.

It follows from (11) and the orthogonality relations for modular group characters that

$$q_{\alpha\beta} = (1/g) \sum_i g_i \psi_\alpha(\mathfrak{K}_i)\bar{\psi}_\beta(\mathfrak{K}_i) = \sum_{\rho,\sigma} b_{\alpha\rho} c_{\rho\sigma} b_{\beta\sigma} \tag{12}$$

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[10] If a linear combination ξ with integral rational coefficients of the ordinary characters $\chi_1, \chi_2, \cdots$ of $\mathfrak{G}$ vanishes for all p-singular elements of $\mathfrak{G}$, then a consideration of ranks shows that ξ can be written in the form $\xi = \sum h_i \Phi_i$ with complex coefficients h_i. The orthogonality relations for modular group characters yield $h_i = (1/g)\sum \xi(R^{-1})\phi_i(R)$ where $\phi_1, \phi_2, \cdots$ are the modular irreducible characters of $\mathfrak{G}$ and where R ranges over all p-regular elements of $\mathfrak{G}$. Each $\phi_i(R)$ can be written as a linear combination of the $\chi_j(R)$ with integral rational coefficients. Substituting this expression for $\phi_i(R)$, we easily see that the h_i are rational integers.

where the $c_{\rho\sigma}$ are the Cartan invariants of $\mathfrak{G}$. The matrix with the coefficients $q_{\alpha\beta}$ is equal to BCB' where $B = (b_{\alpha\beta})$, $C = (c_{\alpha\beta})$. The corresponding quadratic form is equivalent to the form with the matrix C. This yields

THEOREM 6. *If the characters ω_σ^* in Theorem 5 are known, a quadratic form can be found which is equivalent to the form whose matrix is the Cartan matrix of $\mathfrak{G}$ for p.*

Consider an element ξ of Ω,

$$\xi = \sum x_\sigma \psi_\sigma.$$

It follows from (12) that

$$(1/g) \sum g_i \xi(\mathfrak{K}_i) \bar{\xi}(\mathfrak{K}_i) = \sum_{\rho,\sigma} x_\rho x_\sigma q_{\rho\sigma}. \tag{13}$$

If the expression (13) is equal to 1, then $\pm\xi$ is an irreducible ordinary character of $\mathfrak{G}$. Since ξ vanishes for all p-singular classes of $\mathfrak{G}$, its degree is divisible by the highest power p^a of p which divides g.[11] Conversely, if ξ is an irreducible ordinary character of $\mathfrak{G}$ whose degree is divisible by p^a, then ξ vanishes for p-singular classes and belongs, therefore, to Ω. If we set $\xi = \Sigma x_\sigma \psi_\sigma$, the coefficients x_σ give a solution of

$$\sum x_\rho x_\sigma q_{\rho\sigma} = 1.$$

We thus have

THEOREM 7. *Let p^a be the highest power of p which divides the order g of $\mathfrak{G}$. The number of ordinary irreducible representations of $\mathfrak{G}$ whose degree is divisible by p^a is equal to the number of representations of 1 by the quadratic form in Theorem 6. (We count $x_1, x_2, \cdots, x_w$ and $-x_1, -x_2, \cdots, -x_w$ as the same representation.)*

The number determined in Theorem 7 can also be characterized as the number of blocks of defect 0 of $\mathfrak{G}$ (for p). To some extent, Theorem 7 fills a gap left in the investigation of the blocks of a given group.[12]

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[11] See the paper quoted in [9].

[12] R. Brauer, *Proceedings of the National Academy of Sciences*, vol. 30 (1944), pp. 109-114, vol. 32 (1946), pp. 182-186 and 215-219.

PRINCIPAL SOLUTIONS OF DIFFERENCE EQUATIONS.*

By WALTER STRODT.[1]

PART I. Introduction.

One of the central problems in the theory of difference equations is that of eradicating, as far as possible, the arbitrary elements in the manifold of solutions.[2] In brief, a difference equation has so *many* solutions, (in those cases where the complete manifold has been determined, involving one or more arbitrary periodic functions), that the property of a function of being a solution of a given difference equation is of little value in the investigation of the behavior of the function. In order that a function may be studied by means of a difference equation which it satisfies, it is essential in many cases that the function be singled out as playing a somehow distinguished role in the manifold of solutions of that equation.

The concept of principal solution introduced by N. E. Nörlund [3] serves, in the cases of those equations for which the concept has been defined, to distinguish a unique solution, or a unique several-parameter family of solutions. Usually the solutions so distinguished are the most interesting solutions of the equation, being, roughly speaking, the solutions of minimal rate of growth at infinity.[4]

The core of Nörlund's work on principal solutions is the investigation of the two types of equation

$$f(x+\omega)-f(x)=\omega\phi(x), \tag{1}$$

for which the principal solution is defined [5] to be the solution given by the formula

* Received February 7, 1947.

[1] National Research Fellow, at Harvard University; on leave from Columbia University.

[2] Cf. R. D. Carmichael, "The present state of the difference calculus and the prospect for the future," *American Mathematical Monthly*, vol. 31 (1924), pp. 172 ff.

[3] N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Berlin, 1924, Chapters III and IV.

[4] The concept of minimal rate of growth has never been made sufficiently precise to serve as a *definition* of principal solution. It will be noted that in both Nörlund's definition, and the author's given below, appeal is made to other considerations.

[5] Nörlund, *loc. cit.*, pp. 40-43.

$$(2)\qquad f(x) = \operatorname*{limit}_{\eta\to 0} \left[\int_a^\infty \phi(z) e^{-\eta\lambda(z)} dz - \omega \sum_{s=0}^{\infty} \phi(x+s\omega) e^{-\eta\lambda(x+s\omega)} \right]$$

(where a is an arbitrary constant, and $\lambda(z)$ is a function of the form $x^p (\log x)^q$, $(p \geq 1,\ q \geq 0)$), and

$$(3)\qquad f(x+\omega) + f(x) = 2\phi(x)$$

for which the principal solution is defined to be the solution given by the formula

$$(4)\qquad f(x) = \operatorname*{limit}_{\eta\to 0} \left[2 \sum_{s=0}^{\infty} (-1)^s \phi(x+s\omega) e^{-\eta\lambda(x+s\omega)} \right].$$

Carmichael has remarked [6] that in the related problem of *q-difference* equations the elimination of the arbitrary elements is a comparatively easy matter. Where for *difference* equations the crucial question seems to be asymptotic behavior at infinity, and this is difficult to make precise in a general situation, for *q-difference* equations the simple property of analyticity at $x = 0$, (or at $x = \infty$), serves to characterize the distinguished solutions.

For the purpose of studying difference equations we shall find it useful to generalize this concept of distinguished solutions of q-difference equations. We define a *special solution* of a q-difference equation to be a solution expansible in ascending, *not necessarily integral,* powers of x.[7]

Using this concept of *special solution* of q-difference equations we give in this paper a new definition of principal solution, for difference equations with analytic coefficients. In brief, the principal solution of a difference equation is in this paper defined as a solution embedded in *special solutions* of certain q-difference equations which formally approximate the given difference equation. This definition is directly applicable to both linear and non-linear equations, and is effective in singling out those solutions of the difference equation which are of minimal rate of growth at infinity. It affords a vast generalization of Nörlund's results, (in the case of analytic coefficients), and serves to link the somewhat disparate definitions (2) and (4) of Nörlund.

The concepts used below are, in part, modifications of concepts introduced by the author in a paper on non-linear difference equations.[8]

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[6] Carmichael, *loc. cit.*, p. 173.

[7] Cf. Part II, Section C, below.

[8] "Analytic solutions of non-linear difference equations," *Annals of Mathematics*, vol. 44 (1943), pp. 375-396.

PART II. Definitions and Notations.

A. *Quasi-neighborboods of a point.* Let b be any non-zero number. By the *quasi-neighborhood of b, of radius R, and argument γ,* will be meant the set $\mathcal{S}$ consisting of all points x such that $0 < |x - b| < R$, and $\arg(x - b) \neq \gamma$. In other words, a quasi-neighborhood is a circular neighborhood cut along a radius.

B. *Generalized power-series.* Let b be any complex number. By a *generalized power-series* at b is meant a series of the form

$$\sum_{k=0}^{\infty} c_k (x-b)^{\lambda_k}$$

where $0 \leq \lambda_k < \lambda_{k+1}$, $(k = 0, 1, \cdots)$, and $\lim_{k\to\infty} \lambda_k = \infty$, the series being convergent throughout a quasi-neighborhood of b. (We understand by $(x-b)^{\lambda_k}$ the function $e^{\lambda_k \log(x-b)}$, where in the quasi-neighborhood $0 < |x - b| < R$, $\arg(x-b) \neq \gamma$, the branch of the logarithm is to be the one such that $\gamma - \pi < \mathcal{J}(\log(x-b)) \leq \gamma + \pi$.)

C. *Special solutions of q-difference equations.* Let $T_1 x, \cdots, T_n x$ be analytic functions of the complex variable x, such that for some (finite) complex number b the "fixed-point equations" $T_k b = b$, $(k = 1, \cdots, n)$, are all valid. Let $f(x, y_1, \cdots, y_n)$ be a polynomial in the indeterminates y_k with coefficients analytic at $x = b$. By a *special solution* $y(x)$ of the functional equation

$$f(x, y(T_1 x), \cdots, y(T_n x)) = 0 \tag{5}$$

is meant any generalized power-series at b which satisfies (5) in a quasi-neighborhood of b, or an analytic continuation of such a generalized power-series, the continuation being along a radius $\arg(x - b) = \text{constant}$.

Included in this definition is the definition of the special solution of a q-difference equation, in which $T_k x = q_k x$ for some constants q_k, $(k = 1, \cdots, n)$, and in which $b = 0$.

D. *$Q(\alpha)$-sequences.* Let $q_1, q_2, \cdots$ be a sequence of complex numbers, none of which equals unity, the limit of the sequence being unity. Let $b_1, b_2, \cdots$ be a sequence of complex numbers, such that $\lim_{t\to\infty} b_t(1 - q_t) = 1$. If there exists a number α with $-\pi < \alpha \leq \pi$ such that $\arg b_t - \alpha = O(|b_t|^{-1})$, (where O is the Landau order symbol), we shall say that the sequence (q_t, b_t), $(t = 1, 2, \cdots)$, is a *$Q(\alpha)$-sequence.*

E. *Approximating q-difference equations.* Let

$$(6) \qquad f(x, y(x+\omega_1), \cdots, y(x+\omega_n)) = 0$$

be a difference equation, in which the ω_k are given complex numbers, and $f(x, y_1, \cdots, y_n)$ is a polynomial in the indeterminates y_k with coefficients analytic in a region $\mathcal{S}$ which has the property that whenever x_0 is in $\mathcal{S}$, so also are all the points $x_0 + \omega_1, \cdots, x_0 + \omega_n$. Let (q_t, b_t), $(t = 1, 2, \cdots)$, be a $Q(\alpha)$-sequence for some α. Let $q_{tj} = (q_t)^{\omega_j}$, where by $(q_t)^{\omega_j}$ is meant $e^{\omega_j \operatorname{Log} q_t}$.[9] Let $V_{tj} = b_t(1 - q_{tj})$, $(j = 1, \cdots, n;\ t = 1, 2, \cdots)$. Then the sequence of functional equations

$$(7) \qquad f(x, y(q_{t1}x + V_{t1}), \cdots, y(q_{tn}x + V_{tn})) = 0,$$

$(t = 1, 2, \cdots)$, will be called an *approximating sequence of q-difference equations.*[10] Any equation in the sequence (7) will be referred to as an *approximating q-difference equation.*

F. *Primary solutions of difference equations.* Given the difference equation (6), and a function $y(x)$ which is analytic in a region $\mathcal{T}$ having the property that whenever x is in $\mathcal{T}$, so also are all points $x + \omega_j$, $(j = 1, \cdots, n)$. Let α be a number such that $-\pi < \alpha \leq \pi$. We shall say that $y(x)$ is a *primary solution of* (6), *for the region* $\mathcal{T}$, *in the direction* α, (briefly, a $Q(\alpha, \mathcal{T})$ solution of (6)), if there exists a $Q(\alpha)$-sequence (q_t, b_t), $(t = 1, 2, \cdots)$, such that if $y_t(x)$, $(t = 1, 2, \cdots)$, is chosen suitably as a special solution of the approximating q-difference equation (7), then the sequence $y_t(x)$, $(t = 1, 2, \cdots)$, converges to $y(x)$ in $\mathcal{T}$, uniformly in every closed bounded subset of $\mathcal{T}$. (As is customary, we allow in this definition that at each point x of $\mathcal{T}$ finitely many of the functions $y_t(x)$ may fail to be defined.)

G. *Principal solutions of difference equations.* Given the difference equation (6), and a function $y(x)$ which is analytic in a region $\mathcal{U}$ such that whenever x is in $\mathcal{U}$ so also are all the points $x + \omega_1, \cdots, x + \omega_n$. We shall say that $y(x)$ is a *principal solution of* (6), *for the region* $\mathcal{U}$, *in the direction* α, (briefly, a $P(\alpha, \mathcal{U})$ solution of (6)), if for every closed bounded

[9] Throughout this paper we shall use the notation Log z, with upper case, L, to indicate that determination of log z such that $-\pi < \mathfrak{J}\,(\operatorname{Log} z) \leq \pi$. Similarly, we shall use the notation Arg z, with upper case A, to indicate that determination of arg z such that $-\pi < \operatorname{Arg} z \leq \pi$.

[10] It is readily verified that limit $V_{tj} = w_j$, $(j = 1, \cdots, n)$, so that (7) tends formally to (6) as $t \to \infty$, and it is easy to see that (7) is a q-difference equation in the variable $x - b_t$. (Cf. Lemma I, of the Appendix.)

subset $\mathcal{J}_0$ of $\mathcal{U}$, and every positive number ϵ, there exists an ordered set of complex numbers $z_1, \cdots, z_n$ such that $(\sum_{k=1}^{n} | z_k - 1 |^2)^{1/2} < \epsilon$ and such that for some $Q(\alpha, \mathcal{U})$ solution $y(x, z_1, \cdots, z_n)$ of the *parametrized equation*

$$f(x, z_1 y(x+\omega_1), \cdots, z_n y(x+\omega_n)) = 0 \tag{8}$$

the inequality $| y(x, z_1, \cdots, z_n) - y(x) | < \epsilon$ is valid for every x in $\mathcal{J}_0$.

(Obviously every $Q(\alpha, \mathcal{U})$ solution of (6) is a $P(\alpha, \mathcal{U})$ solution of (6). However, the converse is not true. For example, it is almost obvious that the equation $y(x+1) - y(x) = \phi(x)$ has no $Q(\alpha, \mathcal{U})$ solution whatever if $\phi(x)$ is analytic and different from zero at ∞, but for such a condition on $\phi(x)$ this equation has, for every α with $| \alpha | < \pi/2$ and every choice of $\mathcal{U}$ as a half-plane $\mathcal{R}(x) > D > 0$, (with D sufficiently large), a one-parameter family of $P(\alpha, \mathcal{U})$ solutions coinciding with the Nörlund principal solutions.)

PART III. General Program.

The definitions given above are meaningful for every algebraic difference equation (6) with coefficients analytic in a reasonably extensive region. Moreover, special solutions of the approximating q-difference equation

$$f(x, z_1 y(q_{t1} x + V_1), \cdots, z_n y(q_{tn} x + V_n)) = 0, \tag{9}$$

(used for calculating the primary solutions of (8)), are, for most equations (6) and most choices of $z_1, \cdots, z_n$, readily calculated by recursive relations similar to those appearing in the standard procedure for calculating the Taylor's series coefficients at an ordinary point of an algebraic function. However, the remaining convergence discussions, namely the establishment of a limit as t becomes infinite, and after that the establishment of a limit as the z's approach unity, has been carried through in this paper only for certain classes of difference equations, treated in Parts V, VI, and VII. These classes are: (Part V) a broad class of linear and non-linear equations, greatly generalizing part of the author's paper cited above, and including in particular most algebraic difference equations with coefficients analytic at ∞; (Part VI) all equations of the types appearing in Nörlund's "Differenzenrechnung," §§ 32-36, constituting the core of the Nörlund theory of the principal solution (in the analytic case); (Part VII) a simple type of linear equation, for which the variation of the principal solution with the direction is studied.

Part IV of this paper is devoted to a few simple general theorems, used in the sequel, on the relative invariance of the $P(\alpha, \mathfrak{J})$ solution under linear transformation of the independent variable, and under translation of the span.

Part VIII is an appendix in which are collected the definition of a special class of functions, "almost constant functions," useful in Part V, and certain lemmas which are used in Parts V, VI, and VII.

PART IV. On the Relative Invariance of the $P(\alpha, \mathfrak{J})$ Solutions, under Linear Transformations of the Independent Variable, and under Translations of the Span.

THEOREM 1. *Given the difference equation*

$$f(x, y(x+\omega_1), \cdots, y(x+\omega_n)) = 0. \tag{10}$$

Let $s = Ax + B$, *where* A *and* B *are any complex numbers, with* $A \neq 0$. *Let* $h(s) = y(x)$, *let* $\nu_k = A\omega_k$, $(k = 1, \cdots, n)$, *and let*

$$g(s, h(s, \nu_1), \cdots, h(s+\nu_n)) = 0 \tag{11}$$

be the difference equation in $h(s)$ *corresponding to* (10). (*That is,*

$$g(s, h_1, \cdots, h_n) \equiv f((s-B)/A, h_1, \cdots, h_n).)$$

Then if $y_0(x)$ *is a* $P(\alpha, \mathfrak{J})$ *solution of* (10), *the function* $h_0(s)$ *defined by* $h_0(s) \equiv y_0((s-B)/A)$ *is a* $P(\beta, \mathfrak{U})$ *solution of* (11), *where* $\beta \equiv \alpha + \operatorname{Arg} A \pmod{2\pi}$, *and* $\mathfrak{U}$ *is the set of points* s *such that* $(s-B)/A$ *is in* $\mathfrak{J}$.

Proof. Let (q_t, b_t), $(t = 1, 2, \cdots)$, be a $Q(\alpha)$-sequence, and let $y(x, t)$ be a special solution of

$$f(x, z_1 y(q^{\omega_1}x + V_1), \cdots, z_n y(q^{\omega_n}x + V_n)) = 0, \tag{12}$$

where $V_k = b(1 - q^{\omega_k})$, $(k = 1, \cdots, n)$.[11]

Let $h(s, t) = y((s-B)/A, t)$. Then $h(s, t)$ is a special solution of

$$g(s, z_1 h(r^{\nu_1}s + W_1), \cdots, z_n h(r^{\nu_n}s + W_n)) = 0, \tag{13}$$

where $r = q^{(1/A)} = e^{(1/A)\operatorname{Log} q}$, and $W_k = (Ab + B)(1 - r^{\nu_k})$, $(k = 1, \cdots, n)$. Since $(r_t, Ab_t + B)$, $(t = 1, 2, \cdots)$, is a $Q(\beta)$-sequence (cf. Lemma VI of the Appendix), it follows easily that $y_0((s-B)/A)$ is a $P(\beta, \mathfrak{U})$ solution of (11).

[11] Whenever it is convenient we omit the subscript t on q_t, b_t, V_{kt}, etc.

THEOREM 2. *Given the difference equation* (10). *Let c be any complex number. Let $H(x) \equiv y(x-c)$. Let* (10) *be written in the form*

$$f(x, H(x+c+\omega_1), \cdots, H(x+c+\omega_n)) = 0, \tag{14}$$

which will be considered as a difference equation in the unknown function $H(x)$, the spans being $c+\omega_1, \cdots, c+\omega_n$. Let $y_0(x)$ be a $P(\alpha, \mathfrak{J})$ solution of (10). *Then $H_0(x) \equiv y_0(x-c)$ is a $P(\alpha, \mathfrak{U})$ solution of* (14), *where $\mathfrak{U}$ consists of the points x such that $x-c$ is in $\mathfrak{J}$.*

Proof. Let (q_t, b_t) be a $Q(\alpha)$-sequence, and let $y(x,t)$ be a special solution of (12). Let $H(x,t) \equiv y(X_1(x,t), t)$, where

$$X_1(x,t) = q^{-c}x + b(1-q^{-c}). \tag{15}$$

Then $H(x,t)$ is a special solution of

$$f(x, z_1 H(q^{(\omega_1+c)}x + W_1), \cdots, z_n H(q^{(\omega_n+c)}x + W_n)) = 0, \tag{16}$$

where $W_k = b(1-q^{(\omega_k+c)})$, $(k=1, \cdots, n)$. Evidently, $\lim_{t\to\infty} H(x,t) = \lim_{t\to\infty} y(x-c, t)$, and from this the theorem follows at once.

PART V. A Class of Linear and Non-Linear Equations.

THEOREM 3. *Given the difference equation* (10), *with $f(x, y_1, \cdots, y_n)$ a polynomial of degree d in the indeterminates y_k, the coefficients being functions of x analytic at ∞,*[12] *and the ω's being complex numbers, with $\omega_1 = 0$, and $\mathfrak{R}(\omega_k) > 0$ when $k > 1$. Let the polynomial in the y_k obtained from $f(x, y_1, \cdots, y_n)$ by substituting for each coefficient its limit at ∞ be denoted by $F(y_1, \cdots, y_n)$. We shall assume that the following algebraic equation in one unknown γ:*

$$F(\gamma, \cdots, \gamma) = 0 \tag{17}$$

has exactly d distinct finite roots $\gamma_1, \cdots, \gamma_d$, and that for each root γ_j, $(j=1, \cdots, d)$, the "limiting separant function"

$$\sum_{k=1}^{n} F_{y_k}(\gamma_j, \cdots, \gamma_j)\sigma^{\omega_k} \tag{18}$$

is different from zero for every σ in the closed interval $[0,1]$.[13] (*By σ^{ω_k} we understand $e^{\omega_k \operatorname{Log} \sigma}$ if $\sigma \neq 0$, 0 if $\sigma = 0$ and $\omega_k \neq 0$, 1 if $\sigma = 0$ and $\omega_k = 0$.*)

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[12] The condition of analyticity at ∞ is relaxed in the next theorem.

[13] Most equations with coefficients analytic at ∞ fulfill these conditions: in fact, if $f(x, y_1, \cdots, y_n)$ is any polynomial in the y_k, of degree d, with d distinct finite roots

For every positive D let $\mathfrak{H}(D)$ be the half-plane $\Re(x) > D$. Then for every sufficiently large positive D, the $P(0, \mathfrak{H}(D))$ solutions of (10) *consist of exactly d distinct functions $y_j(x)$, $(j = 1, \cdots, d)$, each analytic and bounded in $\mathfrak{H}(D)$; also* $\lim_{\Re(x) \to +\infty} y_j(x) = \gamma_j$, $(j = 1, \cdots, d)$; *finally, every solution of* (10) *which is analytic and bounded in a right half-plane $\mathfrak{H}(D_1)$ is for some positive D_2 coincident in $\mathfrak{H}(D_2)$ with a $P(0, \mathfrak{H}(D_2))$ solution.*

Proof. We first seek the $Q(0, \mathfrak{H}(D))$ solutions of the parametrized equation (8). We shall denote an ordered n-tuple $(z_1, \cdots, z_n)$ by the symbol Z, the ordered n-tuple $(1, \cdots, 1)$ by the symbol Z_0. By the *distance* between $Z = (z_1, \cdots, z_n)$ and $Z' = (z'_1, \cdots, z'_n)$, written $\delta(Z, Z')$, will be meant $(\sum_{k=1}^{n} | z_k - z'_k |^2)^{1/2}$. We shall say that the sequence $Z', Z'', \cdots$ approaches Z as a limit if $\lim_{n \to \infty} \delta(Z^{(n)}, Z) = 0$.

It follows from a straightforward continuity argument that there exist positive numbers ϵ, L, such that if $\delta(Z, Z_0) < \epsilon$, then (a) the equation (19) $F(z_1 C, \cdots, z_n C) = 0$ has exactly d distinct finite roots $C_1, \cdots, C_d$, and (b) for every $Q(0)$-sequence S: (q_t, b_t), $(t = 1, 2, \cdots)$, there is a positive T (depending upon Z and S), such that when $t > T$ the equation

$$f(b_t, z_1 c_0, \cdots, z_n c_0) = 0 \tag{20}$$

has exactly d distinct finite roots c_0, and for each such c_0

$$\left| \sum_{k=1}^{n} z_k f_{y_k}(b, z_1 c_0, \cdots, z_n c_0) q_t^{\lambda \omega_k} \right| > L \tag{21}$$

for every $\lambda \geqq 0$. We shall denote the set of ordered n-tuples Z such that $\delta(Z, Z_0) < \epsilon$ by $\mathfrak{N}$.

We consider (8) for Z in $\mathfrak{N}$. Let S: (q_t, b_t), $(t = 1, 2, \cdots)$ be a $Q(0)$-sequence, and let (12) be the corresponding approximating q-difference equation. Let

$$y(x) = \sum_{\lambda} c_\lambda u^\lambda, \text{ where } u = x - b, \tag{22}$$

$\gamma_1, \cdots, \gamma_d$ for the corresponding $F(\gamma, \cdots, \gamma)$, and with $F_{y_n}(\gamma_j, \cdots, \gamma_j)$ and $F_{y_1}(\gamma, \cdots, \gamma_j)$ both different from zero, $(j = 1, \cdots, d)$, and if $w_1 = 0$ while $w_2, \cdots, w_{n-1}$ are chosen in any fashion as points in the right half-plane, then it is evident that there are d analytic curves in the right half-plane, whose equations can easily be written out explicitly, such that if w_n is chosen as any point of the right half-plane not on any one of these curves, then the corresponding difference equation (10) will satisfy all the hypotheses of the theorem.

be a generalized power-series, if there is one, satisfying (12). Let

$$(23)\qquad G(u) \equiv f(u+b, z_1 \sum_\lambda c_\lambda u^\lambda q^{\omega_1\lambda}, \cdots, z_n \sum_\lambda c_\lambda u^\lambda q^{\omega_n\lambda}).$$

Then $G(u) \equiv 0$. Evidently c_0 is determined by equating to zero the coefficient of u^0 in $G(u)$; it is any root of equation (20), which for t sufficiently large has d distinct finite roots. We assert that if t is sufficiently large, no non-negative real λ exists for which $c_\lambda \neq 0$ and $\lambda \not\equiv 0 \pmod 1$. For, assuming the contrary, let I be an infinite subsequence of the positive integers, such that for each t in I there exists a non-negative real λ such that $c_\lambda \neq 0$ and $\lambda \not\equiv 0 \pmod 1$, and let $\lambda_0 = \lambda_0(t)$ be for each t in I the smallest such λ. Then the coefficient of u^{λ_0} in $G(u)$ is $c_{\lambda_0} \sum_{k=1}^{n} z_k f_{y_k}(b, z_1 c_0, \cdots, z_n c_0) q^{\omega_k \lambda_0}$, and this must be zero for each t in I, in contradiction with the properties of $\mathfrak{N}$. Thus

$$(24)\qquad y(x) = \sum_{\lambda=0}^{\infty} c_\lambda u^\lambda, \text{ an ordinary power-series,}$$

if there is any generalized power-series at all which satisfies (12).

For further study of the c_λ, $(\lambda = 1, 2, \cdots)$, we write $f(x, y_1, \cdots, y_n)$ in the form

$$(25)\qquad f(x, y_1, \cdots, y_n) \equiv \sum_{p=1}^{m} a_p(x) \prod_{s=1}^{s(p)} y_{p,s} - \phi(x),$$

where (for each pair p, s) $y_{p,s}$ is one of the y_k, $(k = 1, \cdots, n)$, and $1 \leq s(p) \leq d$, and $a_p(x)$ and $\phi(x)$ are analytic at ∞. Then

$$(26)\qquad f(x, z_1 y(q^{\omega_1} x + V_1), \cdots, z_n y(q^{\omega_n} x + V_n))$$
$$\equiv \sum_{p=1}^{m} a_p(x) \prod_{s=1}^{s(p)} z_{p,s} y(q_{p,s} x + V_{p,s}) - \phi(x),$$

where $z_{p,s} = z_k$, $\omega_{p,s} = \omega_k$, and $V_{p,s} = V_k$, if $y_{p,s} = y_k$, and where $q_{p,s} = q^{\omega_{p,s}}$. Hence

$$(27)\qquad f(x, z_1 y(q^{\omega_1} x + V_1), \cdots, z_n y(q^{\omega_n} x + V_n))$$
$$\equiv \sum_{p=1}^{m} a_p(x, Z) \prod_{s=1}^{s(p)} y(q_{p,s} x + V_{p,s}) - \phi(x),$$

where $a_p(x, Z) = a_p(x) \prod_{s=1}^{s(p)} z_{p,s}$.

If t is sufficiently large, then (21) holds, and in the new notation this becomes

$$(28) \qquad \Big| \sum_{p=1}^{m} a_p(b, Z) c_0^{s(p)-1}(q^{\lambda}_{p,1} + q^{\lambda}_{p,2} + \cdots + q^{\lambda}_{p,s(p)}) \Big| > L,$$

valid for all non-negative real λ.

Let

$$(29) \qquad a_p(x, Z) = \sum_{\lambda=0}^{\infty} a_{p,\lambda} u^{\lambda}, \qquad (p = 1, \cdots, m),$$

and let

$$(30) \qquad \phi(x) = \sum_{\lambda=0}^{\infty} \phi_{\lambda} u^{\lambda}.$$

Then

$$(31) \qquad \sum_{p=1}^{m} \Big(\sum_{\lambda=0}^{\infty} a_{p,\lambda} u^{\lambda} \Big) \prod_{s=1}^{s(p)} \Big(\sum_{\lambda=0}^{\infty} c_{\lambda} q^{\lambda}_{p,s} u^{\lambda} \Big) = \sum_{\lambda=0}^{\infty} \phi_{\lambda} u^{\lambda},$$

from which the c_{λ}, $(\lambda = 1, 2, \cdots)$, are determined by equations of the following form

$$(32) \qquad c_{\lambda} \sum_{p=1}^{m} a_{p,0} c_0^{s(p)-1}(q^{\lambda}_{p,1} + \cdots + q^{\lambda}_{p,s(p)}) = \phi_{\lambda} - H_{\lambda}(c_0, c_1, \cdots, c_{\lambda-1}, q_{p,s}, a_{i\theta}),$$

where H_{λ}, $(\lambda = 1, 2, \cdots)$ is a polynomial, with positive integers for coefficients, in the indicated arguments, (with $i = 1, \cdots, m$; $p = 1, \cdots, m$; $s = 1, \cdots, s(p)$; $\theta = 1, \cdots, \lambda$). By virtue of inequality (28), it is evident that if numbers C_{λ}, $(\lambda = 1, 2, \cdots)$, are defined recursively by the equations

$$(33) \qquad LC_{\lambda} = | \phi_{\lambda} | + H_{\lambda}(| c_0 | , C_1, \cdots, C_{\lambda-1}, 1, | a_{i\theta} |),$$

then (34) $| c_{\lambda} | \leq C_{\lambda}$, $(\lambda = 1, 2, \cdots)$. (We use here the fact, which follows immediately from Lemma IV, that $| q_{p,s} | < 1$.) If we define $C(u)$ to be $\sum_{\lambda=1}^{\infty} C_{\lambda} u^{\lambda}$, then (33) are the determining relations for the coefficients of an analytic function $C(u)$ which vanishes at $u = 0$ and satisfies the equation

$$(34) \qquad LC(u) = \sum_{\lambda=1}^{\infty} | \phi_{\lambda} | u^{\lambda} + \sum_{p=1}^{m} \Big\{ \Big[\sum_{\lambda=1}^{\infty} | a_{p,\lambda} | u^{\lambda} \Big] [| c_0 | + C(u)]^{s(p)} + | a_{p,0} | [(| c_0 | + C(u))^{s(p)} - | c_0 |^{s(p)} - s(p) | c_0 |^{s(p)-1} C(u)] \Big\}.$$

Let ϵ_0 be any positive number. Then there exists a positive number D independent of t such that if t is sufficiently large, and $| u | < | b | - D$, then $| \sum_{\lambda=1}^{\infty} | \phi_{\lambda} | u^{\lambda} | < \epsilon_0$, and $| \sum_{\lambda=1}^{\infty} | a_{p,\lambda} | u^{\lambda} | < \epsilon_0$, $(p = 1, \cdots, m)$. (See Lemmas XVII and XIV, and Definitions I, II, of the Appendix.) Thus if D is a

sufficiently large positive number, (independent of t), then for each sufficiently large t there is a j in the set $(1,\cdots,d)$, such that the coefficients in (34), considered as an equation in $C(u)$, are arbitrarily near the coefficients in the following equation in X:

$$(35)\qquad LX=\sum_{p=1}^{m}A_p(Z)\{[|\gamma_j(Z)|+X]^{s(p)}-|\gamma_j(Z)|^{s(p)} -s(p)|\gamma_j(Z)|^{s(p)-1}X\},$$

where $A_p(Z)=\lim_{x\to\infty}|a_p(x,Z)|$, and $\gamma_j(Z)$, $(j=1,\cdots,d)$, are the d roots C of (19). Since $C(u)$ vanishes at $u=0$, it follows that, throughout the region $|u|<|b|-D$, $C(u)$ is, for D sufficiently large, arbitrarily near the solution $X=0$ of (35). Since the equation in X obtained from (35) by differentiation with respect to X is not satisfied at $X=0$, $C(u)$ is analytic throughout the region $|u|<|b|-D$, if D is sufficiently large.

Hence it follows from (34) and the fact that c_0 is near one or other of the $\gamma_j(Z)$ when t is large, that there exist positive numbers D, M, independent of t and Z, such that whenever Z is in $\mathfrak{N}$, and S is any $Q(0)$-sequence, then when t is sufficiently large the special solutions $\sum_{\lambda=0}^{\infty}c_\lambda u^\lambda$ of the corresponding approximating q-difference equation (12) satisfy the inequality

$$(36)\qquad \left|\sum_{\lambda=0}^{\infty}c_\lambda u^\lambda\right|<M$$

for all u such that $|u|<|b|-D$. Hence if $y(x,Z)$ is any $Q(0,\mathfrak{H}(D_1))$ solution of (8), and if $D_2=\max(D,D_1)$, then

$$(37)\qquad |y(x,Z)|\leq M$$

throughout the half-plane $\mathfrak{H}(D_2)$ (Cf. Lemma XVIII.) Hence if $y(x)$ is any $P(0,\mathfrak{H}(D_1))$ solution of (10), and if $D_2=\max(D,D_1)$, then

$$(38)\qquad |y(x)|\leq M$$

throughout the half-plane $\mathfrak{H}(D_2)$.

Since for every sufficiently large t we have a special solution of (12) satisfying (36) in the region $|u|<|b|-D$, it follows from the standard compactness argument for bounded families of analytic functions that as t becomes infinite on a suitable subsequence I of the positive integers, $\sum_{\lambda=0}^{\infty}c_\lambda(x-b)^\lambda$ approaches a limit for every x in the half-plane $\mathfrak{H}(D)$, uniformly in every closed bounded subset of that half-plane. Such a limit

function $y_0(x,Z)$ is plainly a $Q(0,\mathfrak{H}(D))$ solution of (8). It is majorized in $\mathfrak{H}(D)$ by the constant M, and moreover, for some j in the set $(1,\cdots,d)$ we have

$$\lim_{\mathfrak{R}(x)\to+\infty} y_0(x,Z)=\gamma_j(Z). \tag{39}$$

For let ϵ_0 be any positive number, and let D_1 be a positive number greater than D and so large that $|C(u)|<\epsilon_0$ if $|u|<|b|-D_1$ and t is sufficiently large. Let x_0 be any point such that $\mathfrak{R}(x_0)>D_1$. Let $u_0=x_0-b$. Let T be so large that $|y_0(x_0,Z)-\sum_{\lambda=0}^{\infty}c_\lambda u_0{}^\lambda|<\epsilon_0$ if $t>T$, and so large that $|c_0-\gamma_j(Z)|<\epsilon_0$ if $t>T$ and j is properly chosen and so large that $|C(u)|<\epsilon_0$ if $|u|<|b|-D_1$ and $t>T$. Then

$$|y_0(x_0,Z)-\gamma_j(Z)|\leq|y_0(x_0,Z)-\sum_{\lambda=0}^{\infty}c_\lambda u_0{}^\lambda|$$
$$+|\sum_{\lambda=1}^{\infty}c_\lambda u_0{}^\lambda|+|c_0-\gamma_j(Z)|\leq 3\epsilon_0.$$

Since c_0 can be chosen, for large t, near any desired one of the $\gamma_j(Z)$ it is easy to see that for any desired j a $y_0(x,Z)$ may be obtained such that $y_0(x,Z)$ tends to $\gamma_j(Z)$ as $\mathfrak{R}(x)$ tends to positive infinity. Let $y_j(x,Z)$, $(j=1,\cdots,d)$ be such limit functions $y_0(x,Z)$ with $\lim_{\mathfrak{R}(x)\to+\infty} y_j(x,Z)=\gamma_j(Z)$, $(j=1,\cdots,d)$.

The functions $y_j(x,Z_0)$, $(j=1,\cdots,d)$, are $Q(0,\mathfrak{H}(D))$, hence $P(0,\mathfrak{H}(D))$ solutions of (10), having all the properties of the $y_j(x)$ asserted in the statement of this theorem, except possibly the uniqueness property.

We assert now that there exist d functions $Y_j(x,b)$, $(j=1,\cdots,d)$, defined for all sufficiently large positive b, and for all x satisfying the condition $|x-b|<b-D_2$, for some positive D_2 independent of b, such that for every solution $y^*(x)$ of (10), bounded and analytic in a right half-plane $\mathfrak{H}(D_3)$, there is a j in the set $(1,\cdots,d)$ such that $y^*(x)=\lim_{b\to\infty} Y_j(x,b)$ in some half-plane $\mathfrak{H}(D_4)$, the limit being uniform in every closed bounded subset of $\mathfrak{H}(D_4)$. The justification of this assertion would evidently complete the proof of the theorem.

Let $\mathfrak{B}_1,\cdots,\mathfrak{B}_d$ be disjoint neighborhoods of $\gamma_1,\cdots,\gamma_d$, respectively. Let b_0 be a positive number such that if $b>b_0$, then there is exactly one root c_0 of

$$f(b,c_0,\cdots,c_0)=0 \tag{40}$$

in each $\mathfrak{B}_j$, $(j=1,\cdots,d)$, and for each such c_0

(41) $$\left|\sum_{k=1}^{n} f_{y_k}(b, c_0,\cdots, c_0)(1-b^{-1})^{\lambda\omega_k}\right| > L,$$

for every $\lambda \geqq 0$. (The existence of such a b_0 follows by continuity from the hypotheses of this theorem.)

For all b greater than b_0 we define q as the function $1-b^{-1}$, and we define $Y_j(x,b)$, as that special solution of

(42) $$f(x, y(q^{\omega_1}x+V_1),\cdots, y(q^{\omega_n}x+V_n)=0$$

for which $Y_j(b,b)$ lies in $\mathfrak{B}_j$, $(j=1,\cdots,d)$. By the discussion given above, $Y_j(x,b)$ is almost constant in direction 0. (Cf. Appendix, Def. II.)

Let $y^*(x)$ be any solution of (10), bounded and analytic in a right half-plane $\mathfrak{H}(D_3)$. Let b be a positive number greater than D_3, and greater than b_0. Then $y^*(x)$ is a solution of the q-difference equation

(43) $$f(x, y(q^{\omega_1}x+V_1),\cdots, y(q^{\omega_n}x+V_n))+E(x,b)=0,$$

where

(44) $$E(x,b)=f(x, y^*(x+\omega_1),\cdots, y^*(x+\omega_n)) \\ -f(x, y^*(q^{\omega_1}x+V_1),\ \cdots, y^*(q^{\omega_n}x+V_n)).$$

Let $y(x)=\sum_{\lambda=0}^{\infty} y_\lambda u^\lambda$, where $u=x-b$. Then by (43) we have

(45) $$f(b, y_0,\cdots, y_0)+E(b,b)=0.$$

Now $E(b,b)=f(b, y^*(b+\omega_1),\cdots, y^*(b+\omega_n))-f(b, y^*(b),\cdots, y^*(b))$. Since $y^*(x)$ is bounded in a right half-plane, it follows from Lemma XX, (with $x=b$), that $y^*(b+\omega_k)-y^*(b)$ is small when b is large, $(k=1,\cdots,n)$. Hence $E(b,b)$ is small when b is large. From this it follows that if b is large then y_0 is in the union $\mathfrak{B}_1+\cdots, \mathfrak{B}_d$, and since y_0 varies continuously with b, there is a fixed J in $(1,\cdots,d)$ such that y_0 is in $\mathfrak{B}_J$ for all large b.

Let $Y_J(x,b)=\sum_{\lambda=0}^{\infty} Y_\lambda u^\lambda$. Let $s(x)=y^*(x)-Y_J(x,b)=\sum_{\lambda=0}^{\infty} s_\lambda u^\lambda$. Let $x_k=q^{\omega_k}x+V_k$, $(k=1,\cdots,n)$. Then $f(x, s(x_1)+Y_J(x_1,b),\cdots, s(x_n)+Y_J(x_n,b))+E(x,b)=0$. Since $f(x, Y_J(x_1,b),\cdots, Y_J(x_n,b))=0$, we conclude that $s(x)$ is a solution of the q-difference equation

(46) $$G(x, s(x_1),\cdots,\ s(x_n))=0,$$

where

$$G(x, s_1, \cdots, s_n) \equiv f(x, s_1 + Y_J(x_1, b), \cdots, s_n + Y_J(x_n, b))$$
$$- f(x, Y_J(x_1, b), Y_J(x_n, b)) + E(x, b).$$

Thus $G(x, s_1, \cdots, s_n)$ is a polynomial in the s_k, with coefficients functions of x and b which are almost constant in direction 0. (That $E(x, b)$ is almost constant in direction 0 is proved in Lemma XXII. The almost constancy of the other coefficients follows from Lemma XV.)

We note that s_0 is small when t is large.

Let $\sigma_k = \sum_{\lambda=1}^{\infty} s_\lambda q^{\lambda\omega_k} u^\lambda$, $(k = 1, \cdots, n)$. Then

$$G(x, s(x_1), \cdots, s(x_n)) = \sum_{k=1}^{n} f_{y_k}(b, s_0 + Y_0, \cdots, s_0 + Y_0)\sigma_k \tag{47}$$
$$+ \sum_{k=1}^{n} [f_{y_k}(x, s_0 + Y_J(x_1, b), \cdots, s_0 + Y_J(x_n, b))$$
$$- f_{y_k}(b, s_0 + Y_0, \cdots, s_0 + Y_0)]\sigma_k$$
$$+ \Sigma' H(i_1, \cdots, i_n; x; b)\sigma_1^{i_1} \cdots \sigma_n^{i_n} + K(x, b),$$

where Σ' is a summation over all positive integers $i_1, \cdots, i_n$ such that $2 \leq i_1 + \cdots + i_n \leq d$, where the $H(i_1, \cdots, i_n; x; b)$ and $K(x, b)$ are almost constant in direction 0, and moreover for some fixed D_4, $K(x, b)$ is arbitrarily small if b is sufficiently large, throughout the region $|x - b| < b - D_4$. (Cf. Lemma XXII, and note that

$$K(x, b) = E(x, b) + f(x, s_0 + Y_J(x_1), \cdots, s_0 + Y_J(x_n))$$
$$- f(x, Y_J(x_1), \cdots, Y_J(x_n)).)$$

Thus, if

$$W_k(x, b) = f_{y_k}(x, s_0 + Y_J(x_1, b), \cdots, s_0 + Y_J(x_n, b)), \qquad (k = 1, \cdots, n),$$

we have

$$\sum_{k=1}^{n} f_{y_k}(b, s_0 + Y_0, \cdots, s_0 + Y_0) \sum_{\lambda=1}^{\infty} s_\lambda u^\lambda q^{\lambda\omega_k} = - \sum_{k=1}^{n} (W_k(x, b) \tag{48}$$
$$- W_k(b, b))\sigma_k - \Sigma' H(i_1, \cdots, i_n; x; b)\sigma_1^{i_1} \cdots \sigma_n^{i_n} - K(x, b).$$

We observe that $W_k(x, b)$ is almost constant in direction 0. Let

$$W_k(x, b) = \sum_{\lambda=0}^{\infty} W_{k,\lambda} u^\lambda, \quad H(i_1, \cdots, i_n; x; b) = \sum_{\lambda=0}^{\infty} H_\lambda(i_1, \cdots, i_n) u^\lambda,$$
$$K(x, b) = \sum_{\lambda=0}^{\infty} K_\lambda u^\lambda.$$

Then the s_λ are determined, $(\lambda = 1, 2, \cdots)$, by equations of the form

$$(49)\qquad s_\lambda \sum_{k=1}^{n} f_{y_k}(b, s_0 + Y_0, \cdots, s_0 + Y_0) q^{\lambda \omega_k}$$
$$= - N_\lambda(s\theta q^{\theta \omega_k}; W_{k,r}; H_p(i_1, \cdots, i_n); K_h),$$

$(r, \theta = 1, \cdots, \lambda - 1;\ k = 1, \cdots, n;\ p = 1, \cdots, \lambda - 2;\ h = 1, \cdots, \lambda)$, where N_λ is a polynomial in the indicated arguments, with positive coefficients.

It follows from a straightforward continuity argument that if b is sufficiently large

$$\left| \sum_{k=1}^{n} f_{y_k}(b, s_0 + Y_0, \cdots, s_0 + Y_0) q^{\lambda \omega_k} \right| > L' = L/2$$

for all positive λ. Thus $|s_\lambda| \leq S_\lambda$, where the numbers S_λ, $(\lambda = 1, 2, \cdots)$ are defined recursively by

$$(50)\qquad L' S_\lambda = N_\lambda(S_\theta; |W_{k,r}|; |H_p(i_1, \cdots, i_n)|; |K_h|).$$

Evidently if $S(u) = \sum_{\lambda=1}^{\infty} S_\lambda u^\lambda$, then $S(u)$ satisfies the algebraic equation

$$(51)\qquad L' S(u) = \sum_{k=1}^{n} (W_k{}^A(x, b) - W_k{}^A(b, b)) S(u)$$
$$+ \Sigma' H^A(i_1, \cdots, i_n; x; b) S^{(i_1 + \cdots + i_n)}(u) + K^A(x, b),$$

where the superscript A indicates here the operation defined by the equation $(\sum_{\lambda=0}^{\infty} F_\lambda u^\lambda)^A = \sum_{\lambda=0}^{\infty} |F_\lambda| u^\lambda$. Let $\lim_{b \to \infty} H^A(i_1, \cdots, i_n; b; b) = h(i_1, \cdots, i_n)$. If D_4 is large, the coefficients in (51), considered as an equation in $S(u)$, are for all large b and all x satisfying $|x - b| < b - D_4$, arbitrarily near the coefficients of the following equation in X:

$$(52)\qquad L' X = \Sigma' h(i_1, \cdots, i_n) X^{(i_1 + \cdots + i_n)}.$$

Hence, since the equation obtained from (52) by differentiation with respect to X is not satisfied when $X = 0$, it follows that if D_4 is fixed as a sufficiently large positive number, then $S(u)$ as defined by (75) and the condition $S(0) = 0$ is analytic in $|u| < b - D_4$, and sufficiently small so that

$$(53)\qquad |\Sigma' H^A(i_1, \cdots, i_n; x; b) S^{(i_1 + \cdots + i_{n-1})}(u)| < L'/4,$$

and

$$(54)\qquad |\Sigma(W_k{}^A(x, b) - W_k{}^A(b, b)| < L'/4, \text{ in } |u| < b - D.$$

Let D_4 be fixed to satisfy these conditions. From (51) we have

$$(55)\qquad S(u) = K^A(x,b)/[L' - \Sigma(W_k{}^A(x) - W_k{}^A(b)) - \Sigma' H^A(i_1,\cdots,i_n;x;b)S^{(i_1+\cdots+i_{n-1})}(u)],$$

so that $|S(u)| < 2K^A(x,b)/L'$ in $|u| < b - D_4$, and since $K^A(x,b)$ is arbitrarily small in $|x - b| < b - D_4$, if b is large, and since s_0 is small when b is large, $|s(x)|$ is small throughout $|x - b| < b - D_4$ if b is large. That is, $y(x) = \lim_{b\to\infty} Y_J(x,b)$, the limit being uniform in every closed bounded subset of the half-plane $\mathfrak{H}(D_4)$. As observed above, this completes the proof of the theorem.

THEOREM 4. *If in the hypotheses of Theorem 3 the phrase "analytic at ∞" is replaced by "almost constant in direction 0," and "limit at ∞" is replaced by "limit as $\mathfrak{R}(x)$ becomes positively infinite," the conclusions remain valid without change, and in addition, for sufficiently large positive D, each $P(0,\mathfrak{H}(D))$ solution is almost constant in direction* 0. (We emphasize the fact, which was used in the proof of Theorem 3, and is proved in Lemma XVII, that every function which is analytic at ∞ is almost constant in direction 0. The $P(0,\mathfrak{H}(D))$ solutions of equation (10) obtained in this paper constitute a large class of functions which are almost constant in direction 0 and which are usually *not* analytic at ∞.)

Proof. The proof is essentially identical with the proof of Theorem 3, since the salient property of functions analytic at ∞ which was used in the proof of Theorem 3 is the property of being almost constant in direction 0.

PART VI. The Nörlund Equations.

The equations to be considered are (1) and (3), under the following assumptions:

Either (56) $\phi(x)$ is an entire function, and for some positive numbers C, K the inequality $|\phi(x)| < Ce^{K|x|}$ is valid for all x, K being less than $2\pi/|\omega|$ in the case of equation (1), less than $\pi/|\omega|$ in the case of equation (3);

Or (57) $\phi(x)$ is analytic at every point of a sector $\mathcal{S}$ defined by inequalities $(-\pi/2) \leq M < \arg x < L \leq (\pi/2)$, and analytic on the boundary of $\mathcal{S}$. Also, ω lies in $\mathcal{S}$. Finally, if $\beta_0 = \min(\pi/2, L - \operatorname{Arg}\omega, \operatorname{Arg}\omega - M)$, then there exist positive numbers C, K such that the inequality $|\phi(x)| < Ce^{K|x|}$ is valid for all x in $\mathcal{S}$, K being less than $(2\pi/|\omega|)\sin\beta_0$

in the case of equation (1), less than $(\pi/|\omega|)\sin\beta_0$ in the case of equation (5).

Under these assumptions the Nörlund principal solution exists in $\mathcal{S}$ and has a representation (for equation (1))

$$f(x) = (\pi/2i)\int_{C_0}\psi(x+\omega z)\csc^2\pi z\,dz, \tag{58}$$

and has a representation (for equation (3)),

$$f(x) = -i\int_{C_0}\phi(x+\omega z)\csc\pi z\,dz, \tag{59}$$

where $\psi(x)=\int_\alpha^x\phi(z)\,dz$, ($\alpha$ being an arbitrary number in the open interval $(-1,0)$, and the path of integration from α to x being a straight line segment), and where C_0 is the contour, (described in the sense of increasing ρ), $z=\alpha+\rho e^{i\beta_0}(-\infty<\rho\leq 0)$, $z=\alpha+\rho e^{-i\beta_0}$, $(0\leq\rho<\infty)$. We define $\beta_0=\pi/2$ in the case where $\phi(x)$ is entire.)[14]

In what follows we shall prove

THEOREM 5. *If either* (56) *or* (57) *is satisfied, then the* $P(\operatorname{Arg}\omega,\mathcal{S})$ *solutions of* (1) *and* (3) *exist and coincide with the Nörlund principal solutions.*

Proof. Section A. The $P(\operatorname{Arg}\omega,\mathcal{S})$ solutions of equation (1). We change the notation slightly, writing (1) in the form

$$y(x+\omega)-y(x)=\omega\phi(x). \tag{60}$$

Let

$$z_1 y(x+\omega)-z_0 y(x)=\omega\phi(x) \tag{61}$$

be the parametrized equation corresponding to (60). Let

$$(q_t, b_t), \qquad (t=1,2,\cdots), \tag{62}$$

be a $Q(\operatorname{Arg}\omega)$-sequence. Let

$$z_1 y(q^\omega x+V)-z_0 y(x)=\omega\phi(x) \tag{63}$$

be an approximating q-difference equation for (61) and (62). (We omit the subscript t; $V=b(1-q^\omega)$.) Let

---

[14] We have paraphrased Nörlund's statements, for the sake of brevity; the essential content of Nörlund's original statements, as well as of the generalizations which he suggests, is contained in this modified version.

$$(64) \qquad y(x)=\sum_\lambda c_\lambda u^\lambda, \qquad \text{(where } u=x-b\text{)},$$

be a generalized power-series, if there is one, satisfying (63) in a quasi-neighborhood of $x=b$. Let

$$(65) \qquad \phi(x)=\sum_{k=0}^{\infty}\phi_k u^k.$$

Then

$$(66) \qquad z_1\sum_\lambda c_\lambda q^{\omega\lambda}u^\lambda - z_0\sum_\lambda c_\lambda u^\lambda = \omega\sum_{k=0}^{\infty}\phi_k u^k.$$

Hence

$$(67) \qquad c_\lambda(z_1 q^{\omega\lambda}-z_0)=\omega\phi_\lambda$$

for every λ, where we define ϕ_λ to be zero if λ is not a non-negative integer.

We consider next the equation

$$(68) \qquad z_1 q^{\omega\lambda}-z_0=0.$$

By Lemma V the existence for each sufficiently large t of a non-negative λ satisfying (68) necessitates that either $z_0=z_1$, or that

$$(69) \qquad [\operatorname{Arg}(z_0/z_1)+2\pi k_0]/[|\operatorname{Log}(z_0/z_1)+2\pi k_0 i|]=0,$$

and

$$(70) \qquad [\operatorname{Log}|z_0/z_1|]/[|\operatorname{Log}(z_0/z_1)+2\pi k_0 i|]=-1$$

for some integer k_0. Equations (69) and (70) imply, for z_0 and z_1 near 1, that $k_0=0$. Thus $\operatorname{Arg}(z_0/z_1)=0$, and $\operatorname{Log}|z_0/z_1|=-|\operatorname{Log}(z_0/z_1)|$, so that $0<(z_0/z_1)<1$. By Lemma V again, $\operatorname{Arg}\operatorname{Log}(q^\omega)=\pi$, whence $0<q^\omega<1$.

We distinguish several cases, according to whether equation (68) has for every sufficiently large t a non-negative solution in λ. If $z_0=z_1$, equations (67) have no solution unless $\phi(b)=0$. We shall put this case aside temporarily.

First Case. $0<z_0/z_1<1$, and, for every sufficiently large t, $0<q^\omega<1$. Then for each sufficiently large t there is exactly one non-negative λ_0 such that $z_1 q^{\omega\lambda_0}-z_0=0$, namely $\lambda_0=[\operatorname{Log}(z_0/z_1)]/[\operatorname{Log}(q^\omega)]$. Thus if equations (67) have any solution at all in the c_λ, then in that solution c_{λ_0} is arbitrary. (If $\phi_{\lambda_0}\neq 0$, there is no solution; if $\phi_{\lambda_0}=0$, then c_{λ_0} is arbitrary.) The case where there is no solution at all can obviously be avoided by suitable choice of the $Q(\operatorname{Arg}\omega)$-sequence. For example, if $R_1, R_2, \cdots$ is a sequence of numbers such that $0<R_t<1$, and $\lim_{t\to\infty} R_t=1$, with R_t

distinct from every k-th root of z_0/z_1, $(k=1,2,\cdots)$, and if $q_t = R_t^{(1/\omega)} = e^{(1/\omega)\operatorname{Log} R_t}$, then for some sequence $b_1, b_2, \cdots$ the sequence of pairs (q_t, b_t), $(t=1,2,\cdots)$, is a $Q(\operatorname{Arg}\omega)$-sequence (by Lemma VII), and for this choice of (62) the equations (67) evidently do have a solution. We confine our attention to the case where there is a solution in the c_λ for equations (67) (of course the other case makes no contribution to the principal solution). Then $c_k = (\omega\phi_k)/(z_1 q^{\omega k} - z_0)$, (all non-negative k such that $z_1 q^{\omega k} - z_0 \neq 0$), and $c_\lambda =$ an arbitrary constant when $\lambda = \lambda_0$. Hence

$$(71) \qquad y(x) = \omega \Sigma' \phi_k u^k (z_1 q^{\omega k} - z_0)^{-1} + c_{\lambda_0} u^{\lambda_0},$$

(where Σ' indicates the sum for all non-negative integers different from λ_0, which may conceivably be an integer). We shall now transform (71) into an expression for $y(x)$ as a contour integral, following steps analogous to those used by Nörlund. Some of the steps will not be justified unless the impossible restriction $|z_1/z_0| < 1$ is made. However, these steps will be carried through formally, as a heuristic device, and the final expression for $y(x)$ will be shown to be valid.

Let $\beta_0 = \pi/2$ in the case (56). Let $\beta_0 = \min(\pi/2, L - \operatorname{Arg}\omega, \operatorname{Arg}\omega - M)$ in the case (57). In either case let β be such that $\beta < \beta_0$, but at the same time $(2\pi \sin\beta)/|\omega| > K$.

We have, formally, from (71),

$$(72) \qquad \begin{aligned} y(x) &= (-\omega/z_0)\Sigma'\phi_k u^k (1 - [z_1 q^{\omega k}/z_0])^{-1} + c_{\lambda_0} u^{\lambda_0} \\ &= (-\omega/z_0)\Sigma'[\phi_k u^k \sum_{n=0}^{\infty} (z_1 q^{\omega k}/z_0)^n] + c_{\lambda_0} u^{\lambda_0} \\ &= (-\omega/z_0)\sum_{n=0}^{\infty}[(z_1/z_0)^n (\Sigma'\phi_k q^{\omega n k} u^k)] + c_{\lambda_0} u^{\lambda_0} \\ &= (-\omega/z_0)\sum_{n=0}^{\infty}[(z_1/z_0)^n \phi(x_n)] + c_{\lambda_0} u^{\lambda_0}, \end{aligned}$$

where $x_0 = x$ and $x_{n+1} = q^\omega x_n + V$, $(n = 0, 1, \cdots)$. Hence

$$(73) \qquad \begin{aligned} y(x) = (-\omega/z_0)(1/2\pi i) &\int_C (z_1/z_0)^\zeta \phi(q^{\omega\zeta}(x-b)+b)\pi \cot \pi\zeta \, d\zeta \\ &+ (-\omega/z_0)\phi(x) + c_{\lambda_0} u^{\lambda_0},^{15} \end{aligned}$$

where C is the contour (described in the sense of increasing ρ), $\zeta = E - \rho e^{i\beta}$, $(-\infty < \rho \leqq 0)$, $\zeta = E + \rho e^{-i\beta}$, $(0 < \rho < \infty)$, with $0 < E < 1$, and where

[15] Cf. Nörlund, *loc. cit.*, p. 69. We have translated the contours one unit to the right to simplify the problem of keeping the variable of integration within the domain of analyticity of $\phi(x)$.

$(z_0/z_1)^\zeta = e^{\zeta \operatorname{Log}(z_0/z_1)}$. This integral may be transformed in turn (using the methods of Nörlund again), to

$$(74)\quad y(x) = (\omega/z_0)\int_{C_1}(z_1/z_0)^\zeta\phi(q^{\omega\zeta}(x-b)+b)(1-e^{-2\pi i\zeta})^{-1}d\zeta$$
$$+(\omega/z_0)\int_{C_2}(z_1/z_0)^\zeta\phi(q^{\omega\zeta}(x-b)+b)(1-e^{2\pi i\zeta})^{-1}d\zeta$$
$$-(\omega/z_0)\int_{C_3}(z_1/z_0)^\zeta\phi(q^{\omega\zeta}(x-b)+b)\,d\zeta$$
$$+(-\omega/z_0)\phi(x)+c_{\lambda_0}u^{\lambda_0},$$

where C_1 is the contour $\zeta = E + \rho e^{i\beta}$, $(0 \leq \rho < \infty)$, C_2 is the contour $\zeta = E + \rho e^{-i\beta}$, $(0 \leq \rho < \infty)$, and C_3 is the contour $\zeta = E + \rho$, $(0 \leq \rho < \infty)$. Employing the change of variables $\theta = q^{\omega\zeta}(x-b)+b$ in the integral over C_3, and using the fact that c_{λ_0} is arbitrary, one obtains

$$(75)\quad y(x) = (\omega/z_0)\int_{C_1}(z_1/z_0)^\zeta\phi(q^{\omega\zeta}(x-b)+b)(1-e^{-2\pi i\zeta})^{-1}d\zeta$$
$$+(\omega/z_0)\int_{C_2}(z_1/z_0)^\zeta\phi(q^{\omega\zeta}(x-b)+b)(1-e^{2\pi i\zeta})^{-1}d\zeta$$
$$+(z_0\operatorname{Log} q)^{-1}u^{\lambda_0}\int_E^{x'}(\theta-b)^{-\lambda_0-1}\phi(\theta)\,d\theta$$
$$+(-\omega/z_0)\phi(x)+cu^{\lambda_0},$$

where $x' = q^{\omega E}(x-b)+b$, and c is an arbitrary constant. We shall show now that if z, z_1 are sufficiently near $1, 1$, then (75) is meaningful, and obtainable from (71) by analytic continuation, provided x is in $\mathcal{S}$, and t is sufficiently large. In fact, we shall show that for every closed bounded subset $\mathcal{F}$ of $\mathcal{S}$, there is a positive T such that if $t > T$ then (75) is meaningful and obtainable from (71) by analytic continuation, for every x in $\mathcal{F}$.

We consider first whether $q^{\omega\zeta}(x-b)+b$ is in the domain of analyticity of $\phi(x)$. It suffices to consider the case (57). Let $\mathcal{F}$ be a closed bounded subset of $\mathcal{S}$. Since $x' = q^{\omega E}x + b(1-q^{\omega E})$, it is evident that if t is sufficiently large, then x' is arbitrarily near $x+\omega E$ throughout $\mathcal{F}$. Since the set $\mathcal{F}+\omega E$ is at a positive distance from the boundary of $\mathcal{S}$, it follows that if t is sufficiently large then x' is in $\mathcal{S}$ for every x in $\mathcal{F}$. Now on C_1 and C_2 $q^{\omega\zeta}(x-b)+b = q^{\omega\rho(\cos\beta\pm i\sin\beta)}(x'-b)+b$, so since x' is in $\mathcal{S}$, and $0<\beta<\beta_0$, we may apply Lemma XIII to show that $q^{\omega\zeta}(x-b)+b$ is in $\mathcal{S}$, if ζ is on C_1 or C_2.

Now the integrals in (75), for any fixed β such that $(2\pi\sin\beta)/|\,\omega\,| > K$,

converge if z_1 and z_2 are sufficiently near unity. (We note that $\phi(q^{\omega\zeta}(x-b)+b$ is bounded when ζ is on C_1 and C_2, since $q^{\omega\zeta}(x-b)+b$ tends to b as a limit when ζ becomes infinite on C_1 or C_2.) We assert that if z_1, z_0 are sufficiently small $y(x)$ as defined by (75) coincides in a quasi-neighborhood of b with $y(x)$ as defined by (71). It suffices to prove that $y(x)$ as defined by (75) is, in a quasi-neighborhood of b, a generalized power-series satisfying (63), since every such function is given by (71). That (75) is a generalized power-series is immediately apparent. That it defines $y(x)$ as a solution of (63) follows from the fact that $z_1 y(q^{\omega}x+V)-z_0 y(x)$, calculated from (75), is easily seen by a familiar argument in the calculus of residues to be equal to $\omega\phi(x)$.

Thus $y(x)$ as defined by (75) is an analytic continuation of $y(x)$ as defined by (71), and it is plain that for every point of $\mathcal{F}$ the analytic continuation is an analytic continuation along a radius $\arg(x-b)=$ constant.[16] We shall use the symbol $y(x;t;z_0,z_1)$ to denote $y(x)$ as defined by (75). Then $y(x;t;z_0,z_1)$ is a special solution of (63), defined in a region $\mathcal{B}_t$, $(t=1,2,\cdots)$, such that if $\mathcal{F}$ is any closed bounded subset of $\mathcal{S}$, then for all sufficiently large t $\mathcal{F}$ is included in $\mathcal{B}_t$.

We assert that if the arbitrary constant c is specialized as a suitable function of t, then $\lim_{t\to\infty} y(x;t;z_0,z_1)$ exists for every x in $\mathcal{S}$, uniformly in every closed bounded subset of $\mathcal{S}$, and that every limit function $y(x;z_0,z_1)$ is given by the equation

$$
\begin{aligned}
(76)\qquad y(x;z_0,z_1) = {} & (\omega/z_0)\int_{C_1}(z_1/z_0)^{\zeta}\phi(x+\omega\zeta)(1-e^{-2\pi i\zeta})^{-1}d\zeta \\
& + (\omega/z_0)\int_{C_2}(z_1/z_0)^{\zeta}(x+\omega\zeta)(1-e^{2\pi i\zeta})^{-1}d\zeta \\
& + (1/z_0)(z_0/z_1)^{(x/\omega)}\int_E^{x''}(z_0/z_1)^{(-\theta/\omega)}\phi(\theta)\,d\theta \\
& + (-\omega/z_0)\phi(x) + c'(z_0/z_1)^{(x/\omega)},
\end{aligned}
$$

where $x''=x+E\omega$, and c' is arbitrary.

We omit the proof, which is completely straightforward. Estimates of $|q^{\omega\zeta}(x-b)+b|$ and $|(q^{\omega\zeta}(x-b)+b)-(x+\omega\zeta)|$, useful for the details of the verification, are given in Lemmas X and XI. We remark also that one satisfactory method of choosing c is in accordance with the formula $c=c'\,e^{-\lambda_0 \operatorname{Log}(-b)}$.

Thus $y(x;z_0,z_1)$ is a $Q(\operatorname{Arg}\omega,\mathcal{S})$ solution of (61); conversely every $Q(\operatorname{Arg}\omega,\mathcal{S})$ solution of (61), obtained under the First Case, is given by (76).

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[16] Whenever $q^{\omega\zeta}(x-b)+b$ is in $\mathcal{S}$, so is $q^{\omega\zeta}(x'-b)+b$ for every x' on the line segment joining b to x.

We consider next the *Second Case,* where either z_0/z_1, or q^ω for infinitely many t, lies outside the open interval $(0,1)$. Conceivably $Q(\operatorname{Arg}\omega, \mathcal{S})$ solutions may exist, not given by (76). To show that this does not happen, it suffices to observe that in the Second Case the generalized power-series (71), (in this Case possibly deprived of the term $c_{\lambda_0}u^{\lambda_0}$), has an analytic continuation (75) with a suitable, (not necessarily arbitrary), c. (This is verified as in the First Case.) Hence *if* there is a limit for $y(x)$, (as given by (75)), as t becomes infinite, (usually there is no limit, but because of the results of the First Case this is a matter of indifference), then it is given by (76).

Next, let R: (z'_0, z'_1), (z''_0, z''_1), $\cdots$ be a sequence of pairs (z_0, z_1) such that $z_0^{(n)}$ and $z_1^{(n)}$ tend to unity as n becomes infinite. We assert that if R is suitably chosen, (namely, to satisfy the condition $0 < (z_0^{(n)}/z_1^{(n)}) < 1$ for every n), then c' can be chosen as a function of n such that $y(x; z_0^{(n)}, z_1^{(n)})$ tends to a limit function $F(x;R)$, uniformly in every closed bounded subset of $\mathcal{S}$, the limit function being given by

$$
\begin{aligned}
(77) \qquad F(x;R) = \omega \int_{C_1} \phi(x+\omega\zeta)(1-e^{-2\pi i\zeta})^{-1}d\zeta \\
+ \omega \int_{C_2} \phi(x+\omega\zeta)(1-e^{2\pi i\zeta})^{-1}d\zeta \\
+ \omega \int_E^{x+E\omega} \phi(\theta)\,d\theta - \omega\phi(x) + c'',
\end{aligned}
$$

where c'' is an arbitrary constant. This is readily verified, and it is easy to see that conversely if the sequence R is chosen in any fashion so that, for a suitable choice of c' as a function of n, the limit $F(x;R)$ exists in $\mathcal{S}$, then $F(x;R)$ is given by (77). Hence (77) gives precisely the totality of $P(\operatorname{Arg}\omega, \mathcal{S})$ solutions. But equation (77) is a step in Nörlund's work leading to equation (58). (Cf. "*Differenzenrechnung*" page 70, equation (9).)[17] Hence the $P(\operatorname{Arg}\omega, \mathcal{S})$ solutions coincide with the Nörlund principal solutions, for equation (1).

Section B. The $P(\arg\omega, \mathcal{S})$ solutions of equation (3).

The treatment is similar to the treatment of equation (1), but simpler, particularly because there is no arbitrary constant to consider. The $P(\operatorname{Arg}\omega, \mathcal{S})$ solution, which is a unique function, (not a one-parameter family of functions), coincides with the Nörlund principal solution (59).

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[17] The arbitrary constant c'', which has no obvious counterpart in Nörlund's equation (9), is innocuous because Nörlund's principal solution has a (somewhat concealed) additive arbitrary constant, the presence of which is indicated by the arbitrary a appearing in the cited equation (9).

PART VII. A Simple Linear Example Emphasizing the Influence of α upon the Principal Solution in Direction α.

THEOREM 6. *Given the difference equation*

$$y(x) - 2y(x+1) = \phi(x), \tag{78}$$

where $\phi(x)$ is analytic at ∞. Let the half-plane $\Re(xe^{-i\alpha}) > D$ be denoted by $\mathfrak{H}(\alpha, D)$. Let $\alpha_k = \mathrm{Arg}(\mathrm{Log}\, 2 + 2\pi ki)$, $(k = 0, \pm 1, \pm 2, \cdots)$. Then the $P(\alpha, \mathfrak{H}(\alpha, D))$ solutions of (78) may be described as follows:

a. *If $\alpha = \alpha_k$, the $P(\alpha, \mathfrak{H}(\alpha, D))$ solution, for D sufficiently large, is of the form $F_k(x) + c\, e^{(\mathrm{Log}(1/2) + 2\pi ki)x}$, where $F_k(x)$ is analytic and for some positive constant M satisfies the inequality $|F_k(x)| < M|x|$ in $\mathfrak{H}(\alpha, D)$, and where c is an arbitrary constant.*

b. *If $\alpha \neq \alpha_k$, the $P(\alpha, \mathfrak{H}(\alpha, D))$ solution, for D sufficiently large, is a uniquely determined function $F(x;\alpha)$, analytic and bounded in $\mathfrak{H}(\alpha, D)$.*

c. *In all cases, every solution $y(x)$ of (78) which in a half-plane $\mathfrak{H}(\alpha, D_1)$ is analytic and satisfies for some positive constants M_1, N the inequality $|y(x)| < M_1|x|^N$, is for some positive D_2 coincident in $\mathfrak{H}(\alpha, D_2)$ with a $P(\alpha, \mathfrak{H}(\alpha, D_2))$ solution.*[18]

Proof. Let C be such that $\phi(x)$ is analytic in the set $|x| \geq C > 0$.

Case 1. Let α be different from every α_k, and let $\cos\alpha > 0$. Let $X = xe^{-i\alpha}$. Then, because of Theorem 1, a necessary and sufficient condition for $y_0(x)$ to be a $P(\alpha, \mathfrak{H}(\alpha, D))$ solution of (78) is that the function $Y_0(X)$ defined by $Y_0(X) \equiv y_0(Xe^{i\alpha})$ be a $P(0, \mathfrak{H}(0, D))$ solution of

$$Y_0(X) - 2Y(X+\omega) = \psi(X), \tag{79}$$

where $\omega = e^{-i\alpha}$ and $\psi(X) \equiv \phi(Xe^{i\alpha})$.

By means of Lemma V it is readily checked that the hypotheses of Theorem 3 are satisfied, so that the $P(0, \mathfrak{H}(0, D))$ solution of (79) is unique and bounded, if D is sufficiently large, and therefore the $P(\alpha, \mathfrak{H}(\alpha, D))$ solution of (78) is unique and bounded, if D is sufficiently large. Now let $y(x)$ be any solution of (78), analytic and satisfying a condition $|y(x)| < M_1|x|^N$ in a half-plane $\mathfrak{H}(\alpha, D_1)$. Let $D_2 \geq D_1$, and $D_2 \geq D$. Let $F(x;\alpha)$ be the $P(\alpha, \mathfrak{H}(\alpha, D))$ solution of (78). Then $y(x) - F(x;\alpha)$

[18] The exponential functions $e^{(\mathrm{Log}(1/2)+2\pi ki)x}$ appearing in the $P(\alpha_k, \mathfrak{H}(\alpha_k, D))$ solutions are bounded in $\mathfrak{H}(\alpha_k, D)$.

is a solution of the homogeneous equation $y(x+1)-2y(x)=0$, and satisfies an inequality $|y(x)-F(x;\alpha)|<M_2|x|^N$ in $\mathfrak{H}(\alpha,D_2)$, whence by Lemma XXIV, $y(x)-F(x;\alpha)=0$ in $\mathfrak{H}(\alpha,D_2)$.

Case 2. Let $\cos\alpha<0$. Let $X=xe^{-i\alpha}$. Then, because of Theorems 1 and 2, a necessary and sufficient condition for $y_0(x)$ to be a $P(\alpha,\mathfrak{H}(\alpha,D))$ solution of (78) is that the function $Y_0(X)$ defined by $Y_0(X)\equiv y_0(Xe^{i\alpha}+1)$ be a $P(0,\mathfrak{H}(0,D-\cos\alpha))$ solution of

$$Y_0(X+\omega)-2Y_0(X)=\psi(X), \tag{80}$$

where $\omega=-e^{-i\alpha}$ and $\psi(X)=\phi(Xe^{i\alpha})$. We now apply Theorem 3, and Lemma XXIV, as in Case 1.

Case 3. Let $\cos\alpha=0$. (This case cannot be brought under Theorem 3, in the manner used for Cases 1 and 2, since after such rotation of the variable we do not have the condition $\mathfrak{R}(\omega)>0$ satisfied.)

We note that if q_t, $(t=1,2,\cdots)$, is a sequence of complex numbers, of positive imaginary part, and *of modulus unity,* and if the sequence tends to unity as a limit, then numbers b_t, $(t=1,2,\cdots)$, can be found such that (q_t,b_t), $(t=1,2,\cdots)$, will be a $Q(\pi/2)$ sequence, and $(1/q_t,-b_t)$, $(t=1,2,\cdots)$ will be a $Q(-\pi/2)$ sequence. (Cf. Lemmas II and VI.)

Let

$$z_0y(x)-2z_1y(x+1)=\phi(x) \tag{81}$$

be the parametrized equation for (78). Let (q_t,b_t), $(t=1,2,\cdots)$, be a $Q(\alpha)$-sequence chosen to satisfy the additional condition $|q_t|\geq 1$. Let

$$z_0y(x)-2z_0y(qx+V)=\phi(x) \tag{82}$$

be the corresponding approximating q-difference equation. Let $y(x;t;z_0,z_1)$ be a special solution of (82). Then if $\phi(x)=\sum_{k=0}^{\infty}\phi_ku^k$, where $u=x-b$, evidently

$$y(x;t;z_0,z_1)=\sum_{k=0}^{\infty}\phi_ku^k/(z_0-2z_1q^k). \tag{83}$$

If z_0,z_1 are both near unity, then $|z_0-2z_1q^k|>1/2$, since $|q^k|\geq 1$, $(k=0,1,\cdots)$. Hence

$$\begin{aligned}|y(x;t;z_0,z_1)|&\leq 2\sum_{k=0}^{\infty}|\phi_k|\;|u^k|\leq 2\sum_{k=0}^{\infty}|u|^kMC/(|b|-C)^{k+1}\\&=2MC/(|b|-C-|u|).\end{aligned}$$

(Cf. Lemma XVI.) Thus, if $|x-b| < |b| - C_1$, with $C_1 > C$, we have $|y(x;t;z_0,z_1)| < 2MC/(C_1 - C)$, from which, by the compactness theorem for bounded families of analytic functions, we conclude the existence of at least one $Q(\alpha, \mathfrak{H}(\alpha, C))$ solution $y(x;z_0,z_1)$ of (81), for all z_0, z_1 such that $2|z_1| - |z_0| > 1/2$, bounded in every half-plane $\mathfrak{H}(\alpha, C_1)$ with $C_1 > C$, and also conclude the existence of at least one $P(\alpha, \mathfrak{H}(\alpha, C))$ solution $y(x)$ of (78), likewise bounded in every half-plane $\mathfrak{H}(\alpha, C_1)$ with $C_1 > C$.

We wish to prove next that every $Q(\alpha, \mathfrak{H}(\alpha, D))$ solution of (81), for $D > C$, is majorized in $\mathfrak{H}(\alpha, D)$ by $2MC/(D-C)$. Let $y^*(x;z_0,z_1)$ be for some positive D a $Q(\alpha, \mathfrak{H}(\alpha, D))$ solution of (81). Let (q_t, b_t), $(t = 1, 2, \cdots)$, be a $Q(\alpha)$ sequence, and let $y(x;t;z_0,z_1)$ be a special solution of (82), such that $y^*(x;z_0,z_1) = \lim_{t\to\infty} y(x;t;z_0,z_1)$, the limit being uniform in every closed bounded subset of $\mathfrak{H}(\alpha, D)$. By virtue of the discussion just above, for the case $|q_t| \geq 1$, we may and do confine our attention to the case where $|q_t| < 1$ for every t.

Let $\lambda_0 = \lambda_0(t)$ be the positive number such that $|z_0| - 2|z_1| \, |q|^{\lambda_0} = 0$. Then

$$y(x;t;z_0,z_1) = \Sigma' \phi_k u^k (z_0 - 2z_1 q^k)^{-1} + c u^{\lambda_0}, \tag{84}$$

where Σ' is the sum over all non-negative integers different from λ_0, and c is a function of t, constant with respect to x.

Let $\psi_t = \operatorname{Arg} q_t$, $L_t = \operatorname{Log} |q_t|$. Then $\lim_{t\to\infty} L_t/\psi_t = 0$, by Lemma II. Let $\epsilon_t = L_t/\psi_t$.

Let $\sigma_t = e^{i\psi_t}$. Then (σ_t, b_t), $(t = 1, 2, \cdots)$, is a $Q(\alpha)$ sequence. Let $h(x;t;z_0,z_1)$ be a special solution of

$$z_0 h(x) - 2z_1 h(\sigma x + W) = \phi(x), \tag{85}$$

where $W = b(1-\sigma)$. Then for some subsequence J of the positive integers, (which we may and do suppose to be the entire sequence of positive integers), $h(x;t;z_0,z_1)$ approaches a limit function $h(x;z_0,z_1)$ as t becomes infinite on J, the limit being uniform in every closed bounded subset of $\mathfrak{H}(\alpha, C)$; moreover, $|h(x;z_0,z_1)| < 2MC/(C_1 - C)$ in every half-plane $\mathfrak{H}(\alpha, C_1)$, with $C_1 > C$. (All this follows, since $|\sigma| = 1$, from the earlier discussion where $|q_t| \geq 1$.) Now

$$h(x;t;z_0,z_1) = \sum_{k=0}^{\infty} \phi_k u^k (z_0 - 2z_1\sigma^k)^{-1}, \tag{86}$$

so that

$$\begin{aligned} &y(x;t;z_0,z_1) - h(x;t;z_0,z_1) \\ &\quad = 2z_1 \Sigma' \phi_k u^k (q^k - \sigma^k)(z_0 - 2z_1 q^k)^{-1}(z_0 - 2z_1\sigma^k)^{-1} + c_1 u^{\lambda_0} \end{aligned} \tag{87}$$

for some constant c_1, (depending on t). We take $C_1 > C$, and $D_4 > D_3 > C_1$, with $D_4 - D_3 < (C_1 - C)/2$, take x_0 so that $D_3 < \Re(x_0 e^{-ia}) < D_4$, and define $u_0 = x_0 - b$. Then

$$(88) \qquad y(x;t;z_0,z_1) - h(x;t;z_0,z_1)$$
$$= 2z_1 u^{\lambda_0} \Sigma' \phi_k (q^k - \sigma^k)(z_0 - 2z_1 q^k)^{-1}(z_0 - 2z_1\sigma^k)^{-1}(u^{k-\lambda_0} - u^{k-\lambda_0})$$
$$+ c_2 u^{\lambda_0},$$

for some constant c_2, (depending upon t). Let $H(x;t;z_0,z_1)$ be the right-hand member of (88), deprived of the term $c_2 u^{\lambda_0}$. Then it is easy to see that if $D_3 < \Re(xe^{-ia}) < D_4$, then $|H(x;t;z_0,z_1)| < M_1 |x - x_0|$ for some positive M_1. (To verify this we note that (a) if $k < \lambda_0/2$, then

$$|z_0 - 2z_1 q^k| > A > 0, \qquad |q^k - \sigma^k| < 2, \qquad |z_0 - 2z_1\sigma^k| > A_1 > 0.$$
$$|u|^{\lambda_0} |u^{k-\lambda_0} - u_0^{k-\lambda_0}| < 2(|b| - C_1)^k, \text{ and } |\phi_k| < MC/(|b| - C)^{k+1},$$

while (b) if $\lambda_0/2 \leq k \leq \lambda_0 + 1$, then

$$|z_0 - 2z_1 q^k| > A_2 |q|^{2k} |\lambda_0 - k| (1 - |q|), \text{ (with } A_2 > 0),$$
$$|q^k - \sigma^k| < k|q - \sigma| = k(1 - |q|), \qquad |z_0 - 2z_1\sigma^k| > A_1 > 0,$$
$$|u^{\lambda_0}| < (|b| - D_3)^{\lambda_0},$$
$$|u^{k-\lambda_0} - u_0^{k-\lambda_0}| < |\lambda_0 - k| \, |x - x_0| (|b| - D_4)^{k-1-\lambda_0},$$

and finally (c) if

$$\lambda_0 + 1 < k, \text{ then } |u^{\lambda_0}| < (|b| - D_3)^{\lambda_0}, \ |q^k - \sigma^k| < k(1 - |q|),$$
$$|z_0 - 2z_1 q^k| > A_3 (1 - |q|)(k - \lambda_0), \qquad |z_0 - 2z_1\sigma^k| > A_4,$$
$$|u^{k-\lambda_0} - u_0^{k-\lambda_0}| < (|b| - D_3)^{k-\lambda_0-1} |x - x_0| (k - \lambda_0).)$$

Hence for some subsequence J_1 of the positive integers, $\lim\limits_{\substack{t\to\infty \\ (t \in J_1)}} H(x;t;z_0,z_1)$ exists in $D_3 < \Re(xe^{-ia}) < D_4$, uniformly in every closed bounded subset of the strip. It follows that in that strip $c_2 u^{\lambda_0}$ approaches a limit as t becomes infinite on J_1, uniformly in every closed bounded subset of that strip. By Lemma XXIII this limit of $c_2 u^{\lambda_0}$ must be identically zero in the strip, since λ_0/b is easily seen to become infinite with t. Thus $|y(x;z_0,z_1) - h(x;z_0,z_1)| < M_2 |x - x_0|$ in the strip $D_3 < \Re(xe^{-ia}) < D_4$. But this implies that $y^*(x;z_0,z_1) - h(x;z_0,z_1) \equiv 0$, since this difference is a solution of the homogeneous equation $z_0 y(x) - 2z_1 y(x+1) = 0$, and no solution of this

which is not identically zero can be majorized by $M_2 \mid x - x_0 \mid$ in the (horizontal) strip $D_3 < \mathfrak{R}(xe^{-ia})\, D_4$. Hence $h(x; z_0, z_1)$ coincides with $y(x; z_0, z_1)$ in the half-plane $\mathfrak{H}(\alpha, D)$, and thus $\mid h(x; z_0, z_1) \mid < MC/(D - C)$ in $\mathfrak{H}(\alpha, D)$.

Plainly it follows from this discussion that every $P(\alpha, \mathfrak{H}(\alpha, D))$ solution of (78) is majorized by $MC/(D - C)$ in $\mathfrak{H}(\alpha, D)$, if $D > C$. The rest of the theorem for this case now follows immediately from Lemma XXIV.

Case 4. $\alpha = \alpha_k$. We first find a particular $P(\alpha, \mathfrak{H}(\alpha, C))$ solution of (78). Let q_t, $(t = 1, 2, \cdots)$, be defined by the equations $\operatorname{Log} \mid q_t \mid = (t + (1/2))^{-1} \operatorname{Log}(1/2)$, $\operatorname{Arg} q_t = 2\pi k/(t + (1/2))$. Let $b_t = (1 - q_t)^{-1}$, $(t = 1, 2, \cdots)$. Then (q_t, b_t), $(t = 1, 2, \cdots)$, is a $Q(\alpha)$-sequence. We note also that $\lim_{t \to \infty} \mid b_t \mid \mid \operatorname{Log} \mid q_t \mid \mid = A_k$, where

$$A_k = (1 + (2\pi k)^2 (\operatorname{Log} 2)^{-2})^{-(1/2)}.$$

Let $y(x, t)$ be a special solution of

$$y(x) - 2y(qx + 1) = \phi(x). \tag{89}$$

Then $y(x, t) = \sum_{k=0}^{\infty} \phi_k u^k (1 - 2q^k)^{-1} + cu^\lambda$, where c is an arbitrary constant and $\lambda = t + (1/2)$. Hence $y(x, t) = u^\lambda \sum_{k=0}^{\infty} \phi_k (u^{k-\lambda} - u_0{}^{k-\lambda})(1 - 2q^k)^{-1} + du^\lambda$, where d is an arbitrary constant. Let $y_1(x, t) = y(x, t) - du^\lambda$. Using estimates similar to those noted in Case 3, we verify easily that for every choice of positive numbers D_1 and D_2, with $D_2 > D_1 > C$, there is a number $M(D_1, D_2)$ such that if $\mathfrak{F}$ is a closed bounded set included in the strip $D_2 > \mathfrak{R}(xe^{-ia}) > D$, then when t is large $\mid y_1(x, t) \mid < M(D_1, D_2) \mid x \mid$ for all x in $\mathfrak{F}$. Hence, by the use of $d = 0$, a $Q(\alpha, \mathfrak{H}(\alpha, C))$ solution $y_0(x)$ of (78) is obtainable, such that $\mid y_0(x) \mid < M(D_1, D_2) \mid x \mid$ in every strip $D_2 > \mathfrak{R}(xe^{-ia}) > D_1 > C$. Of course $y_0(x)$ is also a $P(\alpha, \mathfrak{H}(\alpha, C))$ solution of (78). By using the arbitrariness of d, we readily obtain

$$y_0(x) + c_1 e^{x(\operatorname{Log}(1/2) + 2\pi k i)}$$

as a $P(\alpha, \mathfrak{H}(\alpha, C))$ solution of (78), with c_1 arbitrary.

Now since $\mid y_0(x) \mid \leq M(D_1, D_2) \mid x \mid$ in every strip $D_2 > \mathfrak{R}(xe^{-ia}) > D_1 > C$, it follows from equation (78) itself that for every D greater than C there is a positive number $M(D)$ such that $\mid y_0(x) \mid \leq M(D) \mid x \mid$ in the half-plane $\mathfrak{H}(\alpha, D)$. (We use the relation, obvious from (78), that

$$y_0(x + n) = h_0(x) 2^{-n} + (-1) \sum_{k=0}^{n-1} \phi(x + k) 2^{n-k}.)$$

Next we consider any choice of z_0, z_1 near $1, 1$, and any $Q(\alpha)$-sequence, and following steps similar to those used in obtaining $y_0(x)$, (the essential difference in treatment being that for some choices of z_0, z_1, and some choices of the $Q(\alpha)$-sequence the term with arbitrary coefficient may be lacking), we prove that every $Q(\alpha, \mathfrak{H}(\alpha, D))$ solution of (81) for $D > C$, must be of the form $y_1(x) + c'' e^{x(\mathrm{Log}(z_0/2z_1) + 2\pi k i)}$, where $y_1(x)$ is majorized by an expression of the form $M_2 \mid x \mid$ in $\mathfrak{H}(\alpha, D)$, M_2 being independent of z_0, z_1.

From this it follows that every $P(\alpha, \mathfrak{H}(\alpha, D))$ solution of (78) is of the form $y_2(x) + c''' e^{x(\mathrm{Log}(\frac{1}{2}) + 2\pi k i)}$, where $y_2(x)$ is majorized by $M_2 \mid x \mid$ in $\mathfrak{H}(\alpha, D)$.

The remaining conclusions of the theorem then follow, in Case 4, from Lemma XXIV.

PART VIII. Appendix.

Section A. The fundamental relation between the expansion of $F(x)$ and the expansion of $F(\sigma x + V)$.

LEMMA I. *If* $F(x) = \sum_\lambda F_\lambda (x - b)^\lambda$ *is a generalized power-series at* $x = b$, *convergent in a quasi-neighborhood* $\mathfrak{N}$ *of* b, *and if* σ, V *are complex numbers such that* $b(1 - \sigma) = V$, *then*

$$F(\sigma x + V) = \sum_\lambda F_\lambda \sigma^\lambda (x - b)^\lambda, \tag{90}$$

(*where* $\sigma^\lambda = e^{\lambda \mathrm{Log}\,\sigma}$), *in the quasi-neighborhood* $\mathfrak{N}'$ *consisting of all points* x *such that* $\sigma x + V$ *is in* $\mathfrak{N}$.

Proof. Obvious.

Section B. $Q(\alpha)$-sequences.

LEMMA II. *Let* $q_1, q_2, \cdots$ *be a sequence of complex numbers, each distinct from unity, the limit of the sequence being unity. Let* α *be a number such that* $-\pi < \alpha \leq \pi$. *Then a necessary and sufficient condition that there exist a sequence* $b_1, b_2, \cdots$ *such that the sequence of pairs* (q_t, b_t), $(t = 1, 2, \cdots)$, *be a* $Q(\alpha)$-*sequence is that*

$$\lim_{t \to \infty} (\mathrm{Arg}\,(q_t)) / \mid \mathrm{Log}\, q_t \mid = \sin \alpha, \tag{91}$$

and

$$\lim_{t \to \infty} (\mathrm{Log} \mid q_t \mid) / \mid \mathrm{Log}\, q_t \mid = -\cos \alpha. \tag{92}$$

Proof. Necessity. Since (q_t, b_t), $(t = 1, 2, \cdots)$, is a $Q(\alpha)$-sequence, we must have $b_t \operatorname{Log} q_t = -\theta(t)$, where $\theta(t)$ tends to unity as t becomes infinite. Hence $\operatorname{Arg} \operatorname{Log} q_t \equiv \pi + \operatorname{Arg} \theta(t) - \alpha + \eta(t) \pmod{2\pi}$, where $\eta(t)$ tends to zero as t becomes infinite. This implies $\operatorname{Arg} q_t / |\operatorname{Log} q_t| = \sin(\pi + \operatorname{Arg} \theta(t) - \alpha + \eta(t))$ and $\operatorname{Log} |q_t|) / |\operatorname{Log} q_t| = \cos(\pi + \operatorname{Arg} \theta(t) - \alpha + \eta(t))$, from which the necessity follows at once.

Sufficiency. We define b_t by the equation $b_t = |\operatorname{Log} q_t|^{-1} e^{i\alpha}$. Then $b_t \operatorname{Log} q_t = e^{i(\alpha + \operatorname{Arg} \operatorname{Log} q_t)}$, and since from the hypotheses $\operatorname{Arg} \operatorname{Log} q_t \equiv \pi - \alpha + \eta(t) \pmod{2\pi}$, with $\eta(t)$ tending to zero as t becomes infinite, we have $b_t (\operatorname{Log} q_t)$ tending to -1 as t becomes infinite. Since $\lim_{t \to \infty} (\operatorname{Log} q_t)/(1 - q_t) = -1$, this implies that $\lim_{t \to \infty} b_t (1 - q_t) = 1$.

LEMMA III. *Let (q_t, b_t), $(t = 1, 2, \cdots)$, be a $Q(\alpha)$-sequence. Let ω be a non-zero complex number. Let $\sigma_t = q_t^{\omega}$, where $q_t^{\omega} = e^{\omega \operatorname{Log} q_t}$, $(t = 1, 2, \cdots)$.* Then

$$\lim_{t \to \infty} (\operatorname{Arg} \sigma_t) / |\operatorname{Log} \sigma_t| = \sin(\alpha - \operatorname{Arg} \omega), \tag{93}$$

and

$$\lim_{t \to \infty} (\operatorname{Log} |\sigma_t|) / |\operatorname{Log} \sigma_t| = -\cos(\alpha - \operatorname{Arg} \omega). \tag{94}$$

Proof. Since $\sigma_t = e^{\omega \operatorname{Log} q_t}$, we have $\operatorname{Log} \sigma_t \equiv \omega \operatorname{Log} q_t \pmod{2\pi}$, and since $\operatorname{Log} q_t$ tends to zero as t becomes infinite, this implies that if t is large then $\operatorname{Log} \sigma_t = \omega \operatorname{Log} q_t$. From this relation, and Lemma II, the theorem follows immediately.

LEMMA IV. *If (q_t, b_t), $(t = 1, 2, \cdots)$, is a $Q(\alpha)$-sequence, and ω is a complex number such that* $\cos(\operatorname{Arg} \omega - \alpha) > 0$, *then $|q_t^{\omega}| < 1$ if t is large, and if ω is a complex number such that* $\cos(\operatorname{Arg} \omega - \alpha) < 0$, *then $|q_t^{\omega}| > 1$ if t is large.*

Proof. This follows immediately from Lemma III, equation (94).

LEMMA V. *Let (q_t, b_t), $(t = 1, 2, \cdots)$, be a $Q(\alpha)$-sequence, let ω be a non-zero complex number such that* $\cos(\operatorname{Arg} \omega - \alpha) \neq 0$, *and let ζ be a non-zero complex number. Then a necessary and sufficient condition that there exist for every t a non-negative λ_t such that $q^{\omega\lambda} = \zeta$, (where $q^{\omega\lambda} = e^{\omega\lambda \operatorname{Log} q}$), is that either $\zeta = 1$, or else for some integer k_0*

$$[\operatorname{Arg} \zeta + 2\pi k_0] / |\operatorname{Log} \zeta + 2\pi k_0 i| = \sin(\operatorname{Arg} \omega - 2\alpha), \tag{95}$$

and

(96) $$[\operatorname{Log} |\zeta|]/|\operatorname{Log}\zeta + 2\pi k_0 i| = -\cos(\operatorname{Arg}\omega - \alpha),$$

and for every sufficiently large t.

(97) $$\operatorname{Arg}\operatorname{Log}(q^{\omega}) \equiv \pi + \operatorname{Arg}\omega - \alpha (\operatorname{mod} 2\pi).$$

Proof. Let $\sigma = q^{\omega} = e^{\omega \operatorname{Log} q}$. Then $\sigma^{\lambda} = e^{\lambda \operatorname{Log}\sigma} = \zeta$. Hence $\lambda \operatorname{Log}\sigma = \operatorname{Log}\zeta + 2\pi k(t)i$ for some integer $k(t)$. Thus, if $\lambda \neq 0$, then

(98) $$[\operatorname{Arg}\zeta + 2\pi k(t)]/|\operatorname{Log}\zeta + 2\pi k(t)i| = (\operatorname{Arg}\sigma)/|\operatorname{Log}\sigma|,$$

and

(99) $$(\operatorname{Log}|\zeta|)/|\operatorname{Log}\zeta + 2\pi k(t)i| = (\operatorname{Log}|\sigma|)/(|\operatorname{Log}\sigma|).$$

By (98), (99), and Lemma III, we have

(100) $$\lim_{t\to\infty} [(\operatorname{Arg}\zeta + 2\pi k(t))/|\operatorname{Log}\zeta + 2\pi k(t)i|] = \sin(\alpha - \operatorname{Arg}\omega)$$

(101) $$\lim_{t\to\infty} [(\operatorname{Log}|\zeta|)/|\operatorname{Log}\zeta + 2\pi k(t)i|] = -\cos(\alpha - \operatorname{Arg}\omega).$$

From (101) it follows that $k(t)$ is a constant k_0 for t large. Hence (95) and (96) hold.

Then by (98) and (99) $(\operatorname{Arg}\sigma)/|\operatorname{Log}\sigma| = \sin(\alpha - \operatorname{Arg}\omega)$ and $(\operatorname{Log}|\sigma|)/|\operatorname{Log}\sigma| = -\cos(\alpha - \operatorname{Arg}\omega)$, for all sufficiently large t, and these equations imply (97).

Lemma VI. *If (q_t, b_t), $(t = 1, 2, \cdots)$, is a $Q(\alpha)$-sequence, and ω is a non-zero complex number, then $(q_t^{\omega}, \omega^{-1}b_t)$ is a $Q(\beta)$-sequence, where $\beta \equiv \alpha - \operatorname{Arg}\omega \pmod{2\pi}$.*

Proof. This follows at once from Lemmas II and III, and the easily verified relation $\lim_{t\to\infty} [(\omega^{-1}b_t)(1 - q_t^{\omega})] = 1$.

Lemma VII. *Let $R_1, R_2, \cdots$ be a sequence of numbers such that $0 < R_t < 1$ and $\lim_{t\to\infty} R_t = 1$. Let ω be a non-zero complex number. Let $q_t = R_t^{1/\omega} \equiv e^{(1/\omega)\operatorname{Log} R_t}$, $(t = 1, 2, \cdots)$. Then there is a sequence $b_1, b_2, \cdots$ such that the sequence of pairs (q_t, b_t) is a $Q(\operatorname{Arg}\omega)$-sequence.*

Proof. Evidently $(R_t, (1 - R_t)^{-1})$, $(t = 1, 2, \cdots)$, is a $Q(0)$-sequence. Hence, by Lemma VI, $(R_t^{1/\omega}, (1/\omega)(1 - R_t)^{-1})$, $(t = 1, 2, \cdots)$ is a $Q(\beta)$-sequence, where $\beta \equiv 0 - \operatorname{Arg}(1/\omega) \pmod{2\pi} \equiv \operatorname{Arg}\omega \pmod{2\pi}$.

Section C. Logarithmic spirals.

Lemma VIII. *Let ω be a non-zero complex number. Let (q_t, b_t),*

$(t=1,2,\cdots)$, *be a* $Q(\operatorname{Arg}\omega)$*-sequence. Let* β *be a number such that* $0\leq\beta<(\pi/2)$. *Let* $\xi_1(\rho)=\rho e^{i\beta}$, $\xi_2(\rho)=\rho e^{-i\beta}$ $(0<\rho<\infty)$. *Then if* t *is sufficiently large we shall have* $|q^{\omega\xi_1(\rho)}|\leq 1$, $|q^{\omega\xi_2(\rho)}|\leq 1$ *for all non-negative* ρ.

Proof. Let $\sigma_t=q_t^{\omega}$. Then $q^{\omega\xi_j(\rho)}=\sigma^{\xi_j(\rho)}=e^{\xi_j\operatorname{Log}\sigma}$ $(j=1,2)$. Hence

$$(102)\quad |q^{\omega\xi_j(\rho)}|=\exp\{\mathcal{R}(\xi_j\operatorname{Log}\sigma)\}$$
$$=\exp\{\rho[(\cos\beta)\operatorname{Log}|\sigma|-(\sin\beta)\operatorname{Arg}\sigma]\}$$
$$=\exp\{\rho\cos\beta\operatorname{Log}|\sigma|(1-\tan\beta(\operatorname{Arg}\sigma)/(\operatorname{Log}|\sigma|)\}.$$

By Lemma III, with $\alpha=\operatorname{Arg}\omega$, $\lim_{t\to\infty}[(\operatorname{Arg}\sigma)/(\operatorname{Log}|\sigma|)]=0$. Hence $1-\tan\beta(\operatorname{Arg}\sigma)/(\operatorname{Log}|\sigma|)>0$ if t is large. Also, $\operatorname{Log}|\sigma|<0$ by Lemma IV. Thus $|q^{\omega\xi_j(\rho)}|\leq e^0=1$.

LEMMA IX. *If* $\mathcal{R}(y)<\eta$, *then* $|(1-e^y)/y|\leq e^{|\eta|}$.

Proof. $(e^y-1)/y=\int_0^1 e^{yt}dt$, where the path of integration is a straight line segment. Hence $|(e^y-1)/y|\leq\max_{0\leq t\leq 1}|e^{yt}|=\max_{0\leq t\leq 1}\exp[\mathcal{R}(yt)]$ $=\max_{0\leq t\leq 1}\exp[t\mathcal{R}(y)]\leq e^{|\eta|}$.

LEMMA X. *Under the hypotheses of* Lemma VIII, *let* $\zeta_j=E+\zeta_j(\rho)$, $(j=1,2)$, *where* $0<E<1$. *Then for every positive* δ *there is a positive* T *such that if* $t>T$, *then*

$$|q^{\omega\zeta_j}(x-b)+b|<|x|+(|\omega|+\delta)|\zeta_j|,\qquad (j=1,2).$$

Proof. $$q^{\omega\zeta_j}(x-b)+b=q^{\omega\zeta_j}x+b(1-q^{\omega\zeta_j}).$$

Now $|q^{\omega\zeta_j}|=|q^{\omega E}|\,|q^{\omega\xi_j(\rho)}|$ and each factor is smaller than unity, if t is large (by Lemmas IV and VIII). Thus $|q^{\omega\zeta_j}x|\leq|x|$, $(j=1,2)$, if t is large.

Let $V=b(1-q^{\omega})$. Let δ be any positive number. Let η be any positive number, to be specified later in terms of δ. Let T be so large that if $t>T$, then $|V-\omega|<\eta$, and

$$|[(\omega\operatorname{Log}q)/(e^{\omega\operatorname{Log}q}-1)]-1|<\eta,$$

and

$$|q^{\omega\xi_j(\rho)}|\leq 1,\qquad (j=1,2),$$

for all negative ρ. (See Lemma VIII.) Then $|q^{\omega\zeta_j}|\leq 1$, so that

$$\mathcal{R}(\omega\zeta_j\operatorname{Log}q)\leq 0,$$

and therefore by Lemma IX we have

$$|(1-e^{\omega\zeta_j \operatorname{Log} q})/(\omega\zeta_j \operatorname{Log} q)| \leq 1.$$

Now

$$(103) \qquad b(1-q^{\omega\zeta_j}) = V\frac{(1-q^{\omega\zeta_j})}{(1-q^{\omega})}$$

$$= (V\zeta_j)\cdot\frac{e^{\omega\zeta_j \operatorname{Log} q}-1}{\omega\zeta_j \operatorname{Log} q}\cdot\frac{\omega \operatorname{Log} q}{e^{\omega \operatorname{Log} q}-1}.$$

Hence $|b(1-q^{\omega\zeta_j})| < (|\omega| + \eta)|\zeta_j|(1+\eta)$. If η is sufficiently small, these results imply

$$|q^{\omega\zeta_j}(x-b)+b| < |x| + (|\omega| + \delta)|\zeta_j|, \qquad (j=1,2).$$

LEMMA XI. *Let $\mathfrak{B}_1$, $\mathfrak{B}_2$ be any bounded sets in the x, ζ planes, respectively. Let (q_t, b_t) be a $Q(\alpha)$-sequence. Let ω be a complex number. For every positive δ there exists a positive T such that if $t > T$, then*

$$|[q^{\omega\zeta}(x-b)+b]-[x+\omega\zeta]| < \delta$$

for all x in $\mathfrak{B}_1$ and all ζ in $\mathfrak{B}_2$.

Proof. Let $V = b(1-q^{\omega})$. Then

$$(104) \qquad |[q^{\omega\zeta}(x-b)+b]-[x+\omega\zeta]| = \Big| x(e^{\omega\zeta \operatorname{Log} q}-1) + (V-\omega)\zeta$$
$$+ (V\zeta)\left(\frac{e^{\omega\zeta \operatorname{Log} q}-1}{\omega\zeta \operatorname{Log} q}-1\right) + V\zeta\left(\frac{e^{\omega\zeta \operatorname{Log} q}-1}{\omega\zeta \operatorname{Log} q}\right)\left(\frac{\omega \operatorname{Log} q}{e^{\omega \operatorname{Log} q}-1}-1\right)\Big|.$$

Let $|x| < M_1$ in $\mathfrak{B}_1$, $|\zeta| < M_2$ in $\mathfrak{B}_2$. Let $\delta > 0$. Let $\eta > 0$ be such that $|e^{\gamma}-1| < \delta/(4M_1)$ if $|\gamma| < \eta$, and such that

$$\left|\frac{e^{\gamma}-1}{\gamma}-1\right| < \delta/(4M_2(|\omega|+1))$$

if $|\gamma| < \eta$, and such that

$$\left|\frac{e^{\gamma_1}-1}{\gamma_1}\right|\left|\frac{\gamma_2}{e^{\gamma_2}-1}-1\right| < \delta/(4M_2(|\omega|+1))$$

if $|\gamma_1| < \eta$ and $|\gamma_2| < \eta$. Let T be such that if $t > T$ then all the inequalities $|V-\omega| < 1$, $|V-\omega| < \delta/(4M_2)$, $|\operatorname{Log} q| < \eta/(M_2|\omega|)$, and $|\operatorname{Log} q| < \eta/|\omega|$ are valid.

Then if $t > T$ we have

$$|[q^{\omega\zeta}(x-b)+b]-[x+\omega\zeta]| < M_1[\delta/(4M_1)] + [\delta/(4M_2)]M_2$$
$$+ (|\omega|+1)M_2[\delta/(4M_2(|\omega|+1))]$$
$$+ (|\omega|+1)M_2[\delta/(4M_2(|\omega|+1))]$$
$$= \delta.$$

LEMMA XII. (1) *Let* $f(X) = Pe^{X \cot C} - \sin X$ *where* $P > 0$ *and* $0 < C < \pi/2$. *Then* $f(X) > 0$ *throughout the interval* (X_1, ∞), *where* X_1 *satisfies* $0 < X_1 < \pi$, *provided all the following are valid*:

a. $P > \sin X_1 \, e^{-X_1 \cot C}$

b. $X_1 > C$

c. $P > e^{-(2\pi + C)\cot C} \sin C$.

(2) *Let* $y(Y) = Re^{Y \cot C} + \sin Y$, *where* $R > 0$ *and* $0 < C < \pi/2$. *Then* $g(Y) > 0$ *throughout* (Y_1, ∞), *where* Y_1 *satisfies* $-\pi < Y_1 < 0$, *provided all the following are valid*:

a. $R > | \sin Y_1 | \, e^{-Y_1 \cot C}$

b. $Y_1 > C - \pi$

c. $R > e^{-(\pi + C)\cot C} \sin C$.

Proof by elementary calculus.

LEMMA XIII. *Let* $\mathcal{S}$ *be the sector* $-\pi/2 \leq M < \arg \zeta < L \leq \pi/2$. *Let* ω *be a point of* $\mathcal{S}$. *Let* $\alpha = \operatorname{Arg} \omega$. *Let* $\beta_0 =$ minimum $(\pi/2, L - \alpha, \alpha - M)$. *Let* $0 < \beta < \beta_0$. *Let* $\mathfrak{F}$ *be a closed bounded subset of* $\mathcal{S}$. *Let* (q_t, b_t), $(t = 1, 2, \cdots)$, *be a* $Q(\alpha)$*-sequence. Let* $\zeta_1 = q^{\omega z_1}(x - b) + b$, $\zeta_2 = q^{\omega z_2}(x - b) + b$, *with* $z_1 = \rho(\cos \beta + i \sin \beta)$, $z_2 = \rho(\cos \beta - i \sin \beta)$ *and* $0 \leq \rho \leq \infty$. *Then if* t *is sufficiently large,* ζ_1 *and* ζ_2 *are both in* $\mathcal{S}$ *for every non-negative* ρ *and every* x *of* $\mathfrak{F}$.

Proof. We shall consider only ζ_1, since ζ_2 is treated similarly. Let $\sigma_t = q_t^{\omega}$, $(t = 1, 2, \cdots)$. Then $(\sigma_t, \omega^{-1} b_t)$, $(t = 1, 2, \cdots)$, is a $Q(0)$-sequence. (By Lemma VI). Hence, by Lemma II,

$(\operatorname{Arg} \sigma_t)/| \operatorname{Log} \sigma_t | = \sin \eta_t$, and $(\operatorname{Log} | \sigma_t |)/| \operatorname{Log} \sigma_t | = -\cos \eta_t$, where $\lim_{t \to \infty} \eta_t = 0$. Let $\delta = \delta_t(x) = \operatorname{Log} | 1 - x/b_t |$, and let $\epsilon = \epsilon_t(x) = \operatorname{Arg} (1 - x/b_t)$. It is readily seen that

$$\mathcal{R} (\zeta_1/b) = 1 - e^{-r \cos(\beta - \eta) + \delta} \cos (r \sin(\beta - \eta) - \epsilon) \tag{105}$$

and

$$\mathcal{J} (\zeta_1/b) = e^{-r \cos(\beta - \eta) + \delta} \sin (r \sin (\beta - \eta) - \epsilon) \tag{106}$$

where $r = \rho \, | \operatorname{Log} \sigma_t |$. Thus r varies from 0 to $+\infty$ as ρ varies from 0 to $+\infty$. Let $M_1 = M - \operatorname{Arg} b$, $L_1 = L - \operatorname{Arg} b$. We are to prove that

$$M_1 < \operatorname{Arg} (\zeta_1/b) < L_1. \tag{107}$$

If r is large, $\mathfrak{R}(\zeta_1/b)$ is near 1 and $\mathfrak{J}(\zeta_1/b)$ is near 0. Hence (107) is satisfied if r is large. Hence if (107) fails for some non-negative r, then for some non-negative r either

$$\text{(108)} \qquad \operatorname{Arg}(\zeta_1/b) = L_1$$

or

$$\text{(109)} \qquad \operatorname{Arg}(\zeta_1/b) = M_1.$$

By elementary calculations it is found that (105), (106) and (108) imply that the equation

$$\text{(110)} \qquad Pe^{X\cot C} - \sin X = 0$$

holds for some X in (X_1, ∞), where

$$X_1 = L_1 - \epsilon,\ C = \beta - \eta,\ X = L_1 + r\sin C - \epsilon,$$

and

$$P = \sin L_1 e^{-(L_1-\epsilon)\cot C - \delta},$$

and that (105), (106) and (109) imply that the equation

$$\text{(111)} \qquad Re^{Y\cot C} + \sin Y = 0$$

holds for some Y in (Y_1, ∞), where

$$Y_1 = M_1 - \epsilon,\ C = \beta - \eta,\ Y = M_1 + r\sin C - \epsilon,$$

and

$$R = |\sin M_1|\, e^{-(M_1-\epsilon)\cot C - \delta}.$$

It is readily seen that (110) contradicts Lemma XII (1), while (111) contradicts Lemma XII (2).

Section D. Almost constant functions.

Definition I. Let S: (q_t, b_t), $(t = 1, 2, \cdots)$, be a $Q(\alpha)$-sequence. Let $f(x,t)$ be a function of x and t, which is defined for every sufficiently large positive integer t as a function of x analytic in a neighborhood of b_t, with Taylor's expansion $\sum_{k=0}^{\infty} f_k(t)\,(x - b_t)^k$. The function $f(x,t)$ will be called "*almost constant on the sequence* S" if $f_0(t)$ is a bounded function of t, and if for every positive ϵ there is a positive D, (independent of t), such that for all sufficiently large t the inequality

$$\text{(112)} \qquad \sum_{k=1}^{\infty} |f_k(t)|(|b_t| - D)^k < \epsilon$$

is valid.

Definition II. Let $F(x, b)$ be a function of x and b which, for all sufficiently large b such that $\operatorname{Arg} b = \alpha$, is analytic at $x = b$, with Taylor's expansion $\sum_{k=0}^{\infty} F_k(b)(x - b)^k$. We shall say that $F(x, b)$ is "*almost constant in direction* α" if $F(b, b)$ is a bounded function of b, when $\operatorname{Arg} b = \alpha$, and b is large, and if for every positive ϵ there is a positive D (independent of b), such that for all sufficiently large b the inequality

$$(113) \qquad \sum_{k=1}^{\infty} |F_k(b)|(|b_t| - D)^k < \epsilon$$

is valid.

Lemma XIV. *Let $F(x, b)$ be almost constant in direction α. Let S: (q_t, b_t), $(t = 1, 2, \cdots)$, be a $Q(\alpha)$-sequence. Let $f(x, t)$ be defined as the function $F(x, |b_t| e^{i\alpha})$. Then $f(x, t)$ is almost constant on S.*

Proof. Let $F(x, c) = \sum_{k=0}^{\infty} F_k(c)(x - c)^k$ for all c such that $\operatorname{Arg} c = \alpha$. Let $c_t = |b_t| e^{i\alpha}$. Then $|c_t - b_t| = |b_t| \; |e^{i\alpha} - e^{i \operatorname{Arg} b}| = O(1)$. Let ϵ be a positive number. Let D_1 be such that

$$\sum_{k=1}^{\infty} |F_k(c)|(|c| - D_1)^k < \epsilon$$

for c sufficiently large, with $\operatorname{Arg} c = \alpha$. Let D_2 be such that $|b_t - c_t| < D_2/2$ for all sufficiently large t. Let $D = D_1 + D_2$. Then

$$\begin{aligned} (114) \qquad f(x, t) = F(x, c_t) &= \sum_{k=0}^{\infty} F_k(c_t)(x - c_t)^k \\ &= \sum_{n=0}^{\infty} (x - b_t)^n \sum_{k=n}^{\infty} F_k(c_t) \binom{k}{n} (b_t - c_t)^{k-n} \\ &= \sum_{n=0}^{\infty} (x - b_t)^n f_n(t). \end{aligned}$$

Hence

$$\begin{aligned} (115) \qquad \sum_{n-1}^{\infty} |f_n(t)|(|b_t| - D)^n &\leq \sum_{k=1}^{\infty} |F_k(c_t)|(|b_t| - D + |b_t - c_t|)^k \\ &\leq \sum_{k=1}^{\infty} |F_k(c_t)|(|c_t| - D_1)^k < \epsilon. \end{aligned}$$

Also, since $|f_0(t)| \leq |F(c_t, c_t)| + \sum_{k=0}^{\infty} |F_k(c_t)|(D_2/2)^k$, $f_0(t)$ is bounded. Hence $f(x, t)$ is almost constant on S.

Lemma XV. (1) *Let $f(x, t)$ and $g(x, t)$ be almost constant on a $Q(\alpha)$-sequence S. Then $f(x, t) + g(x, t)$ and $f(x, t) g(x, t)$ are almost constant on S.*

(2) *Let $F(x,b)$, $G(x,b)$ be almost constant in direction α. Then $F(x,b)+G(x,b)$ and $F(x,b)G(x,b)$ are almost constant in direction α.*

Proof. Obvious.

LEMMA XVI. *Let $\phi(x)$ be analytic at ∞, and in the domain $|x|\geq C$. Then there is a positive M such that $|\phi^{(k)}(b)|\leq M\,k!/(|b|-C)^{k+1}$ if $k>0$ and $|b|>C$.*

Proof. Let $|\phi(x)|\leq M_1$ in the set $|x|\geq C$. Then, if $\phi(x)=\sum_{j=0}^{\infty}a_jx^{-j}$, we have $|a_j|\leq M_1C^j$. Let $|b|>C$. Let $u=x-b$. Then

$$\phi(x)=\sum_{j=0}^{\infty}a_j(u+b)^{-j}=\sum_{k=0}^{\infty}u^k\sum_{j=0}^{\infty}a_j\binom{-j}{k}b^{-(j+k)}. \tag{116}$$

Thus

$$|\phi^{(k)}(b)|\leq k!\sum_{j=0}^{\infty}|a_j|\left|\binom{-j}{k}\right||b|^{-(j+k)} \tag{117}$$

$$\leq k!\sum_{j=0}^{\infty}M_1C^j\left|\binom{-j}{k}\right||b|^{-(j+k)}.$$

Let $H(x)=M_1-(M_1C)/(x+C)=M_1x/(x+C)$. Let $v=x+|b|$. Then

$$H(x)=\sum_{k=0}^{\infty}v^k\sum_{j=0}^{\infty}M_1C^j|b|^{-(j+k)}\left|\binom{-j}{k}\right|. \tag{118}$$

Hence $|\phi^{(k)}(b)|\leq H^{(k)}(-|b|)=k!\,M_1C(|b|-C)^{-(k+1)}$, which establishes the formula with $M=M_1C$.

LEMMA XVII. *Every function analytic at ∞ is almost constant in direction α, for every α.*

Proof. This follows immediately from Lemma XVI.

LEMMA XVIII. *Let (q_t,b_t), $(t=1,2,\cdots)$, be a $Q(\alpha)$-sequence. Let C be a positive number. Let $\mathcal{S}_t$ be the point-set $|x-b_t|<\mathfrak{R}(b_te^{-i\alpha})-C$. Then if $\mathcal{J}$ is any closed bounded set such that $\mathfrak{R}(xe^{-i\alpha})>C$ at every x of $\mathcal{J}$, there is a positive T such that $\mathcal{J}$ is included in $\mathcal{S}_t$ whenever $t>T$.*

Proof. Let $\sigma=e^{-i\alpha}$. Then

$$|x-b|^2=|x\sigma-b\sigma|^2=[\mathfrak{R}(\sigma b)-C]^2+\{\mathfrak{R}(\sigma b)\}\cdot \tag{119}$$

$$\{[2C-2\mathfrak{R}(\sigma x)]+[\tan(\operatorname{Arg}b-\alpha)\mathfrak{J}(b\sigma)$$

$$-2\mathfrak{J}(x\sigma)\tan(\operatorname{Arg}b-\alpha)+(|x|^2-C^2)/\mathfrak{R}(\sigma b)]\}.$$

Since $\lim_{t\to\infty}\mathfrak{R}(b\sigma)=\lim_{t\to\infty}|b|\cos(\operatorname{Arg}b-\alpha)=+\infty$, and $\tan(\operatorname{Arg}b-\alpha)$

$= O(|b|^{-1})$, while $\Im(b\sigma) = |b| \sin(\operatorname{Arg} b - \alpha) = O(1)$, it follows from equation (119) that there is a positive T such that if $t > T$, then $|x - b|^2 < (\Re(\sigma b) - C)^2$ for all x in $\mathfrak{J}$.

Lemma XIX. *If $f(x)$ is almost constant in direction α, then $f(x)$ tends to a finite limit as the real part of $xe^{-i\alpha}$ tends to positive infinity.*

Proof. Let ϵ be any positive number. Let D be such that if $|b| > D$, and $\operatorname{Arg} b = \alpha$, and $f(x) = \sum_{k=0}^{\infty} f_k(b)(x-b)^k$, then

$$\sum_{k=1}^{\infty} |f_k(b)|(|b| - D)^k < \epsilon.$$

Let x_1, x_2 be any two complex numbers such that $\Re(x_1 e^{-i\alpha}) > D$ and $\Re(x_2 e^{-i\alpha}) > D$.

Let b be such that $|b| > D$ and $\operatorname{Arg} b = \alpha$. Now

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f(b)| + |f(b) - f(x_2)|$$

$$= |\sum_{k=1}^{\infty} f_k(b)(x_1 - b)^k| + |\sum_{k=1}^{\infty} f_k(x_2 - b)^k| \leq \sum_{k=1}^{\infty} |f_k(b)|\,|x_1 - b|^k$$
$$+ \sum_{k=1}^{\infty} |f_k(b)|\,|x_2 - b|^k < 2\epsilon$$

if b is sufficiently large, since $|x_1 - b| < |b| - D$, and $|x_2 - b| < |b| - D$, if b is sufficiently large (by Lemma XVIII).

Lemma XX. *Let $F(x)$ be bounded and analytic in a right half-plane, $\Re(x) > D > 0$, and on the boundary of that half-plane. Let b be a positive number, and let $q = 1 - 1/b$. Let ω be a fixed complex number of positive real part. Let $V = b(1 - q^\omega)$, where $q^\omega = e^{\omega \operatorname{Log} q}$.*

Let D_1 be any positive number greater than D. Then there exists a positive number M (independent of b and x), such that

$$|F(q^\omega x + V) - F(x + \omega)| < (M|x|)/(b\Re(x))$$

for all sufficiently large b and all x in the circle $|x - b| \leq b - D_1$.

Proof. Let x be any point of the circle $|x - b| \leq b - D_1$. Then $x = b + \rho e^{i\theta}$, where $0 \leq \rho \leq b - D_1$, and $-\pi < \theta \leq \pi$. Hence

$$|\tan \operatorname{Arg} x| = |\rho \sin\theta (b + \rho\cos\theta)^{-1}| \leq (\rho)^{1/2}(2D_1)^{1/2}$$
$$\leq (b)^{1/2}(2D_1)^{-1/2}.$$

Now $\mathfrak{R}(q^\omega x + V) = \mathfrak{R}(q^\omega x) + \mathfrak{R}(V)$. Since V tends to the limit ω as b becomes infinite, it follows that $\mathfrak{R}(V) > 0$ if b is large. Hence $\mathfrak{R}(q^\omega x + V) > \mathfrak{R}(q^\omega x)$ if b is large. Let $q^\omega = Re^{i\psi}$, with $R > 0$ and $-\pi < \psi \leq \pi$. Then $\lim_{b\to\infty} \psi/\operatorname{Log} R = \tan(\operatorname{Arg}\omega)$, by Lemma III. Now $\operatorname{Log} R = O(b^{-1})$, and $\psi = O(b^{-1})$. Hence $\sin\psi = O(b^{-1})$, and $1 - \cos\psi = O(b^{-2})$. We have

$$\begin{aligned}(120)\qquad \mathfrak{R}(q^\omega x) &= R\,|\,x\,|\cos(\psi + \operatorname{Arg} x)\\ &= R\,|\,x\,|\,[\cos\psi\cos(\operatorname{Arg} x)](1 - \tan\psi\tan\operatorname{Arg} x).\end{aligned}$$

Now $\tan\psi\tan\operatorname{Arg} x \leq M_1(b^{-1})(b)^{1/2}$, for some positive M_1 (independent of b and x). Let A be any positive number less than 1. Then it follows from (120) that $\mathfrak{R}(q^\omega x) > A\mathfrak{R}(x)$ if b is sufficiently large. Hence, if b is sufficiently large, and x lies in the circle $|\,x - b\,| < b - D_1$ then $q^\omega x + V$ lies in the half-plane $\mathfrak{R}(x) > D$. Now

$$\begin{aligned}(121)\qquad & F(q^\omega x + V) - F(x + \omega)\\ &= (2\pi i)^{-1}\int_C \frac{F(\zeta)[(q^\omega x + V) - (x + \omega]}{(\zeta - (q^\omega x + V))(\zeta - (x + \omega))}\,d\zeta\end{aligned}$$

where C is any circle lying in the half-plane $\mathfrak{R}(x) \geq D$ and containing $q^\omega x + V$ and $x + \omega$ in its interior. We take C to be the circle with center $x + \omega$ and radius $r = \mathfrak{R}(x + \omega) - D$. Then $q^\omega x + V$ will lie in C provided $|(x + \omega) - (q^\omega x + V)| < r$. There exists a positive number A_1 such that $|(x + \omega) - (q^\omega x + V)| < A_1\,|\,x\,|/b$, for all x, since the relation $q = 1 - 1/b$ implies that $\omega - V = O(b^{-1})$. Hence to show that $q^\omega x + V$ lies in C it suffices to show that $A_1\,|\,x\,|\,/\,b < \mathfrak{R}(x + \omega) - D$. Since there is a positive number A_2 such that $\mathfrak{R}(x + \omega) - D > A_2\mathfrak{R}(x)$ when $\mathfrak{R}(x) \geq D_1$, it suffices to show that $A_1\,|\,x\,|/b < A_2\mathfrak{R}(x)$, or that $b\cos(\operatorname{Arg} x) > A_1/A_2$. Since $\cos(\operatorname{Arg} x) = (1 + \tan^2\operatorname{Arg} x)^{-1/2} > (1 + b/2D_1)^{-1/2}$, when $|\,x - b\,| < b - D_1$, we conclude that if b is sufficiently large, and $|\,x - b\,| < b - D_1$, then $b\cos(\operatorname{Arg} x) > A_1/A_2$. It follows that if b is sufficiently large, and $|\,x - b\,| < b - D_1$, then $q^\omega x + V$ is in C.

By the same argument, if b is sufficiently large, then $|(x + \omega) - (q^\omega x + V)| < r/2$ for all x in the set $|\,x - b\,| < b - D_1$. Then if $|\,F(x)\,| \leq M_1$ in the half-plane $\mathfrak{R}(x) \geq D$, we have

$$\begin{aligned}(122)\qquad & |\,F(q^\omega x + V) - F(x + \omega)\,|\\ &< (2\pi)^{-1}\int_0^{2\pi} \frac{M_1(A_1\,|\,x\,|/b)r\,d\theta}{(r/2)A_2\mathfrak{R}(x)} = (M\,|\,x\,|)/(b\mathfrak{R}(x)).\end{aligned}$$

Lemma XXI. *Let $G(x, b)$ be a function of x and b such that, for some*

positive D_1, $G(x,b)$ *is analytic in the circle* $|x-b|<b-D_1$ *for all sufficiently large positive* b, *and in that circle satisfies an inequality* $|G(x,b)| < M|x|/b\mathfrak{R}(x)$. *Then* $G(x,b)$ *is almost constant in direction* 0, *and moreover, for every* $D_2 > D$, *there is a positive number* M_2 *such that if* $G(x,b)=\sum_{\lambda=0}^{\infty} G_\lambda (x-b)^\lambda$, *then* $|\sum_{\lambda=0}^{\infty} |G_\lambda|u^\lambda| < M_2 b^{-1/4}$ *if* $|u|<b-D_2$.

Proof. Let $G(x,b)=\sum_{\lambda=0}^{\infty} G_\lambda u^\lambda$, where $u=x-b$. Let D_3 be such that $D_2 > D_3 > D_1$. Let $\rho = b - D_3$. If $|u|<b-D_2$, then

$$
\begin{aligned}
(123)\qquad |\sum_{\lambda=0}^{\infty}|G_\lambda|u^\lambda|^2 &= |\sum_{\lambda=0}^{\infty}|G_\lambda|\rho^\lambda(u/\rho)^\lambda|^2 \\
&\leqq (\sum_{\lambda=0}^{\infty}|G_\lambda|^2\rho^{2\lambda})(\sum_{\lambda=0}^{\infty}|u/\rho|^{2\lambda}) \\
&= (1-|u/\rho|^2)^{-1}(2\pi)^{-1}\int_0^{2\pi}|G(b+\rho e^{i\theta}|^2\,d\theta \\
&\leq (1-|u/\rho|^2)^{-1}(2\pi)^{-1}\int_0^{2\pi}\frac{M^2|b+\rho e^{i\theta}|^2}{b^2\mathfrak{R}^2(b+\rho e^{i\theta})}\,d\theta \\
&= (1-|u/\rho|^2)^{-1}M^2b^{-1}(b^2-\rho^2)^{1/2} < M^2(D_2-D_3)^{-1}D_3^{-1/2}b^{-1/2}.
\end{aligned}
$$

LEMMA XXII. *Let* $f(x,y_1,\cdots,y_n)$ *be a polynomial in the* y_k, *with coefficients functions of* x *analytic and bounded in a right half-plane. Let* $Y(x)$ *be a function of* x *analytic and bounded in a right half-plane. Let* $\omega_1,\cdots,\omega_n$ *be complex numbers, with* $\omega_1 = 0$ *and* $\mathfrak{R}(\omega_k) > 0$ *when* $k>1$. *Let* b *be a positive number, let* $q=1-1/b$, *and let* $V_k = b(1-q^{\omega_k})$, $(k=1,\cdots,n)$. *Let*

$$
\begin{aligned}
E(x,b) &= f(x, Y(x+\omega_1),\cdots,Y(x+\omega_n)) \\
&\quad - f(x, Y(q^{\omega_1}x+V_1),\cdots,Y(q^{\omega_n}x+V_n)).
\end{aligned}
$$

Then $E(x,b)$ *is almost constant in direction* 0, *and moreover if* $E(x,b)$ $\sum_{\lambda=0}^{\infty} E_\lambda(x-b)^\lambda$, *then there exists a positive* D_3 *and a positive* M_3, *both independent of* b, *such that*

$$
(124)\qquad |\sum_{\lambda=0}^{\infty}|E_\lambda|u^\lambda| < M_3 b^{-1/4}
$$

if $|u|<b-D_3$ *and* b *is sufficiently large.*

Proof. Directly from its definition $E(x,b)$ is the sum of finitely many terms of the form

$$a(x)\{\prod_{k=1}^{n}[Y(x+\omega_k)]^{i_k}-\prod_{k=1}^{n}[Y(q^{\omega_k}x+V_k)]^{i_k}\} \tag{125}$$

where $a(x)$ is analytic and bounded in a right half-plane, and $i_1,\cdots,i_n$ are non-negative integers such that $i_1+\cdots+i_n\geqq 1$. Such a term may be written in the form

$$a(x)\sum_{s=1}^{M}h_s(x)[Y(x+\omega_{k(s)})-Y(q^{\omega_{k(s)}}x+V_{k(s)})], \tag{126}$$

where $M=i_1+\cdots+i_n$, and $k(s)$ is for each s an integer between 1 and n, and $h_s(x)$ is a power product in the $Y(x+\omega)$ and the $Y(q^{\omega}x+V)$. Thus, since $a(x)$ and each $h_s(x)$ is bounded, it follows from Lemma XX that $E(x,b)$ satisfies an inequality $|E(x,b)|<M_4|x|(b\Re(x))^{-1}$ in some circle $|x-b|<b-D_4$. Hence Lemma XXI is applicable, to establish the present theorem.

Section E. Special lemmas.

LEMMA XXIII. *Let $b_1,b_2,\cdots$ be a sequence of complex numbers tending to ∞. Let $\lambda_1,\lambda_2,\cdots$ be a sequence of non-negative numbers, such that* $\lim_{t\to\infty}(\lambda_t/b_t)=\infty$. Let *$c_1,c_2\cdots$ be any sequence of complex numbers. Let $f(x,t)=c_t(x-b_t)^{\lambda_t}$, in a simply connected region $\mathfrak{B}$ containing none of the points b_t. (Any determination of $(x-b_t)^{\lambda_t}$ being used.)*

Then if $\lim_{t\to\infty} f(x,t)$ *exists, uniformly in an open subset $\mathfrak{B}_0$ of $\mathfrak{B}$, the limit function is identically zero in $\mathfrak{B}_0$.*

Proof. Evidently $f'(x,t)/f(x,t)=\lambda_t/(x-b_t)$, where $f'(x,t)$ is the derivative of $f(x,t)$ with respect to x. Hence $f'(x,t)/f(x,t)$ becomes infinite as t becomes infinite. This precludes the possibility that $f(x,t)$ tend to a non-zero finite limit at a point of $\mathfrak{B}_0$.

LEMMA XXIV. *Let* $\alpha_k=\operatorname{Arg}(\operatorname{Log}2+2\pi ki)$, $(k=0,\pm 1,\pm 2,\cdots)$. *Let $h(x)$ be a solution of the equation*

$$h(x)-2h(x+1)=0, \tag{127}$$

analytic in a half-plane $\Re(xe^{-i\alpha})>D>0$, and satisfying there a condition $|h(x)|\leqq M|x|^N$ for some positive M,N.

Then $h(x)$ is bounded in $\Re(xe^{-i\alpha})>D$, and if α is different from every α_k, then $h(x)\equiv 0$, while if $\alpha=\alpha_k$, then $h(x)=c\,e^{\sigma_k x}$ for some constant c, and for $\sigma_k=\operatorname{Log}(1/2)+2\pi ki$.

Proof. Case 1. $\cos\alpha \neq 0$. In this case equation (127) provides an analytic continuation of $h(x)$ throughout the entire plane. Let $f(x) = 2^x h(x)$. Then $f(x)$ is an entire function of period 1. Hence $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$. Let $z = e^{2\pi i x}$, and let $G(z) = f(x) = \sum_{n=-\infty}^{\infty} a_n z^n$. Then $G(z)$ is analytic for all finite z except $z = 0$. Suppose there is a j such that $a_j \neq 0$. Let $H(z) = z^{-j} G(z) = \sum_{n=-\infty}^{\infty} a_n z^{n-j}$. Let $r = |z|$, and let r be large. Then there is a positive C such that if z is chosen suitably, with $|z|$ large, then $|z^{-j}G(z)| \geq C$. That is, $|G(z)| \geq C r^j$. In other words, if x is chosen suitably, with $|e^{2\pi i x}| = r$, and r large, then $|f(x)| \geq C r^j$.

On the boundary of the half-plane $\mathfrak{R}(xe^{-i\alpha}) > D$ we have $\mathfrak{R}(x) = D \sec\alpha - \mathfrak{I}(x)\tan\alpha$. Let $\mathfrak{I}(x) = -(\operatorname{Log} r)/(2\pi)$. Then $|e^{2\pi i x}| = r$. Hence for some x with $\mathfrak{I}(x) = -(\operatorname{Log} r)/(2\pi)$, we have $|f(x)| \geq C r^j$. Because of the periodicity of $f(x)$, we may take this x to have real part equal to $D \sec\alpha + ((\operatorname{Log} r)(\tan\alpha)/(2\pi)) + \theta$, where $0 < \theta \leq 1$, if $\cos\alpha > 0$, and $-1 \leq \theta < 0$ if $\cos\alpha < 0$. Then x lies in the half-plane $\mathfrak{R}(xe^{-i\alpha}) > D$. Hence

$$|h(x)| \leq M\,|x|^N \leq M_1 (\operatorname{Log} r)^N, \text{ for some positive } M_1. \tag{128}$$

But

$$\begin{aligned} |h(x)| &= |2^{-x}|\,|f(x)| \geq \exp(-\mathfrak{R}(x)(\operatorname{Log} 2))C r^j \\ &\geq C_1 \exp[(\operatorname{Log} r)(j - (\operatorname{Log} 2)(\tan\alpha)/(2\pi))]. \end{aligned} \tag{129}$$

It follows that $j \leq (\operatorname{Log} 2)(\tan\alpha)/(2\pi)$.

By a similar argument, using r sufficiently *small*, one shows that if $a_j \neq 0$, then $j \geq (\operatorname{Log} 2)(\tan\alpha)/(2\pi)$.

Thus, if $a_j \neq 0$, then $j = (\operatorname{Log} 2)(\tan\alpha)/(2\pi)$. Hence $h(x) \equiv 0$, unless there is an integer k such that $\tan\alpha = (2\pi k)/(\operatorname{Log} 2)$; if $\tan\alpha = (2\pi k)/(\operatorname{Log} 2)$, then $h(x) \equiv c\, 2^{-x} e^{2\pi i k x} = c\, e^{\sigma_k x}$, for some constant c.

Now if $\tan\alpha = (2\pi k)/(\operatorname{Log} 2)$, either $\alpha = \alpha_k$, or $\alpha \equiv \alpha_k + \pi \pmod{2\pi}$. For the second possibility we must have $c = 0$, because if $c \neq 0$, then $c\, e^{\sigma_k x}$ is evidently not majorized by $M\,|x|^N$ in the half-plane $\mathfrak{R}(xe^{-i\beta}) > D$ when $\beta \equiv \alpha_k + \pi \pmod{2\pi}$. Thus the lemma follows, in Case 1.

Case 2. $\cos\alpha = 0$. If $h(x_0) = A \neq 0$, then $h(x_0 - n) = 2^n A$, which contradicts the hypothesis $|h(x_0 - n)| \leq M\,|x_0 - n|^N$.

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ON TOPLER'S WAVE ANALYSIS.*

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In a quite forgotten paper [2], appearing in a periodical which was not generally accessible even at the time of its publication (1872), the physicist Töpler [1] has dealt with a generalization of harmonic analysis [2] (on a finite time-range). The generalization in question consists in replacing the overtones of $\sin t$ and/or $\cos t$ by the sequence $\phi(t), \phi(2t), \phi(3t), \cdots$, where $\phi(t)$ is an "arbitrary" periodic function.

As implied by context *and* date, Töpler's considerations on such a basic sequence are purely formal in nature, even though some of his formulae are quite ingenious. From what was available at that time, he could have, perhaps, derived more solid foundations from a suggestion made to him by Boltzmann, which he mentions but does not follow further. Boltzmann's suggestion is that $\phi(t)$ should be expanded into its ordinary Fourier series, in which t should then be replaced by $2t, 3t, \cdots$; thus obtaining a (nonrecursive) system of an infinity of linear equations (to be solved), connecting the functions $\phi(nt)$ with the functions $\sin mt$ and/or $\cos mt$.

This program can be carried out today. In its full generality, which leads to very simple results but without which the theory would become cumbersome, it could not have been carried out at that time. In fact, recourse will have to be had not only to Lebesgue's definition of an integral (without which there is no Fischer-Riesz theorem) but also to Hilbert's theory of bounded matrices. What will be needed in the latter regard is the theory of bounded "D-matrices," introduced by Toeplitz [3]; cf. also the Appendix of reference 2) in [4].

* Received December 10, 1946.

[1] To the mathematician, Töpler's name is familiar in connection with the quadratic extremal property of Fourier constants; a connection in which he is mentioned, for instance, on p. 639 of vol. 2 of Hobson's *Theory of Functions* (2nd ed.).

[2] In [5], I failed to refer to Töpler's paper, which I knew well twenty years earlier but which I had not seen in the meantime. I must have forgotten it, even though I clearly was under its influence (in particular in connection with Boltzmann's suggestion). I realized this only recently, while turning over Burkhardt's report [1].

In the literature known to me, Burkhardt's abstract (*loc. cit.*, pp. 905-906) is the only passage referring to Töpler's paper [2].

As pointed out in [4], p. 147, what is actually accomplished by Toeplitz's Dirichlet matrices is a representation of the Eratosthenian sieve process. Hence, the situation agrees with Töpler's prediction of arithmetical implications ([2], p. 98). On the other hand, it cannot be surprising that certain assertions made, anno 1872, by a physicist interested in wave analysis should prove to be inaccurate (wrong is, for instance, the criterion claimed in [2], p. 71, concerning the sufficiency of absolute convergence).

Actually, the whole problem becomes well-defined only if it is specified whether true *series* (with unique coefficients) or just approximability by *sequences* (that is to say, the "separation" property) is required, and there result two additional questions if the (L^2)-metric is replaced by the l. u. b.-metric of uniform convergence (the corresponding questions concerning mere convergence, or convergence almost everywhere, are of course at least as intricate, hence "special," as in the particular case of ordinary Fourier series). It would be hard to say or, rather, it would be unhistorical to ask, which of these possibilities Töpler and Boltzmann actually had in mind.

With reference to a sequence of constants $\phi_1, \phi_2, \cdots$, let $D(\Phi)$ denote the infinite matrix in which the indices of the rows and the columns range through the positive integers and the n-th column is formed by the sequence

$$(0, \cdots, 0, \phi_1, 0, \cdots, 0, \phi_2, 0, \cdots, 0, \phi_3, \cdots), \tag{1}$$

where each block of 0's is of length $n-1$ (it being understood that these blocks are considered as missing when their length is 0, i. e., in the first column). Since $(\phi_1, \phi_2, \phi_3, \cdots)$ is the first, $(0, \phi_1, 0, \phi_2, 0, \cdots)$ the second, $(0, 0, \phi_1, 0, 0, \phi_2, \cdots)$ the third, $\cdots$ column of $D(\Phi)$, every element of $D(\Phi)$ situated above the principal diagonal is 0 and every element in the principal diagonal is ϕ_1. The n-th component of the vector, say $(f_1, f_2, \cdots)$, into which a vector, say $(c_1, c_2, \cdots)$, is transformed by $D(\Phi)$ is

$$f_n = \sum_{d|n} \phi_d c_{n/d}, \tag{2}$$

where d runs through all divisors $(\geqq 1)$ of n.

In Toeplitz's paper [3], the matrix of his D-form is chosen to be the transposed matrix of the matrix of the linear substitution (2); so that (2) becomes replaced by

$$f_n' = \sum_{m=1}^{\infty} \phi_n c_{nm} \tag{2 bis}$$

(nm is the product of n and m, and not a double subscript). In contrast to

(2), the transposed substitution, (2 bis), is defined for every $(c_1, c_2, \cdots)$ satisfying

$$(3) \qquad \sum_{n=1}^{\infty} | c_n |^2 < \infty$$

if and only if

$$(4) \qquad \sum_{m=1}^{\infty} | \phi_m |^2 < \infty.$$

If (2 bis) is used instead of (2), the Eratosthenian connection becomes obscured. However, since the properties of a matrix with regard to Hilbert's space are identical with those of the transposed matrix, Toeplitz's theory of D-forms can be formulated in terms of (2 bis) as well as in terms of (2).

When expressed in terms of the latter, the Eratosthenian, algorithm, Toeplitz's principal results in [3] are that

(i) the matrix $D(\Phi)$ is bounded (in Hilbert's sense) if and only if the ordinary Dirichlet series

$$(5) \qquad \Phi(s) = \sum_{n=1}^{\infty} \phi_n / n^s$$

is convergent in the half-plane $\sigma > 0$ and satisfies the condition

$$(6) \qquad | \Phi(s) | < \text{const., where } \sigma > 0,$$

$(s = \sigma + it)$; that

(ii) if the assumptions of (i) are satisfied and, in addition,

$$(7) \qquad | 1/\Phi(s) | < \text{Const., where } \sigma > 0,$$

then $D(\Phi)$ has a (unique) bounded reciprocal matrix, $D^{-1}(\Phi)$, which is precisely $D(\Psi)$, where $\Psi(s) = 1/\Phi(s)$; a fact which, as shown in [4], can be refined by saying that

(ii bis) if the assumptions of (i) are satisfied, then the spectrum of $D(\Phi)$ (meant in the sense in which I defined the spectrum of any, not necessarily Hermitian, bounded matrix) is identical with the closure of the values attained by the function (5) in the half-plane $\sigma > 0$.

In addition, use will be made of the following pair of facts, in which A^* denotes $\bar{A}'$, if $\bar{A}$ is the complex conjugate, and A' the transposed matrix, of a matrix A (so that, in particular, $z^* = \bar{z} = x - iy$, if $z = x + iy$ is a number):

(α) If $A^* = A$, where $A = (a_{nm})$, then the Hermitian form

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} c_n c_m^*,$$

when thought of as a double series with the restriction that, in its partial sums, the index pair (n, m) occurs whenever (m, n) occurs, is convergent at every point $(c_1, c_2, \cdots)$ of Hilbert's space if and only if A is bounded.

(β) An arbitrary matrix $A = (a_{nm})$, where $n, m = 1, 2, \cdots$, is bounded if and only if either the product AA^* or the product A^*A (exists and) is bounded.

Both (α) and (β) are classical (Toeplitz, Hellinger-Toeplitz); cf. pp. 121-132 of my *Spektraltheorie*, 1929).

From (β) and (i), it is easy to deduce the following criterion (cf. [5], pp. 575-576), the proof of which will be detailed for the sake of completeness:

(I) *Let $\phi(t)$ be a function of class (L^2) on the t-interval $(0, \pi)$, i. e., let* (4) *be satisfied by the constants $\phi_1, \phi_2, \cdots$ which result by placing*

$$\phi(t) \sim \sum_{n=1}^{\infty} \phi_n \sin nt \tag{8}$$

on $(0, \pi)$. Then the infinite Hermitian matrix

$$\left(\int_0^{\pi} \phi(nt)\phi^*(mt)\,dt\right), \qquad (n, m = 1, 2, \cdots), \tag{9}$$

where $\phi(t)$ is meant to be defined for $-\infty < t < \infty$ by (8), *is a bounded matrix if and only if the Dirichlet series* (5) *is convergent in the half-plane $\sigma > 0$ and satisfies* (6).

Although

$$\int_0^{\pi} \phi(t)\,dt = 0 \tag{10}$$

is not assumed, the summation index $n = 0$ does not occur in (8). It does occur in the corresponding cosine series,

$$\phi(t) \sim \sum_{n=0}^{\infty} \phi_n \cos nt, \tag{11}$$

of $\phi(t)$ on $(0, \pi)$. However, it does not occur in, or does not have any

influence on, the boundedness of either (5) or (9), and it will be clear from the proof of (I) that

(I bis) (8) *can be replaced by* (11) *in* (I), *provided that* (10) *is satisfied* (*i. e., if* $\phi_0 = 0$).

The case of

$$\phi(t) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt), \tag{12}$$

where (instead of $0 < t < \pi$)

$$0 < t < 2\pi, \tag{13}$$

can, of course, also be treated.

According to (8),

$$\phi(nt) \sim \sum_{k=1}^{\infty} \phi_k \sin knt$$

and (z^* denoting the complex conjugate of z)

$$\phi^*(mt) \sim \sum_{k=1}^{\infty} \phi_k^* \sin kmt.$$

Hence, an application of the ("bilinear" form of the "quadratic") Parseval relation shows that the integral occurring in (9) is a fixed multiple of the sum

$$\sum_{k=1}^{\infty} \phi_{km/(n,m)} \phi_{kn/(n,m)}^*, \tag{14}$$

where (n, m) denotes the greatest common divisor of n and m (and divides, therefore, the numerators $n, m, 2n, 2m, \cdots, kn, km, \cdots$ of the indices occurring in this bilinear form of ϕ_i and ϕ_j^*). On the other hand, since $D(\Phi)$ has been defined as the matrix the n-th column of which is the sequence (1), the sum (14) clearly is the complex conjugate matrix element at intersection of the n-th row and the m-th column of the product $D^*(\Phi)D(\Phi)$, the last asterisk being the one defined before (α). It follows therefore from (β) that the matrix (9) is bounded if and only if the matrix $D(\Phi)$ is. Consequently, (I) follows from (i).

It will now be shown that, if (α) is used instead of (β), there results from (I) the following theorem:

(II) *Let* $\phi(t)$ *be a function defined for* $-\infty < t < \infty$ *by* (8), *where* $\phi_1, \phi_2, \cdots$ *is a sequence of constants satisfying* (4). *Then the partial sums,*

$$f^k(t) = \sum_{n=1}^{k} c_n \phi(nt), \tag{16}$$

of the infinite series

(17) $$\sum_{n=1}^{\infty} c_n \phi(nt)$$

belonging to any *sequence* $(c_1, c_2, \cdots)$ *satisfying* (3) *tend in the mean* (L^2) *to a function*

(17 bis) $$f(t) = f(t; c_1, c_2, \cdots),$$

(of class (L^2) *on* $(0, \pi)$*) if and only if the Dirichlet series* (5) *is convergent in the half-plane* $\sigma > 0$ *and satisfies* (6).

In other words, there belongs to every or not to every point $(c_1, c_2, \cdots)$ of Hilbert's space a function $f(t)$ satisfying

(18) $$\int_0^{\pi} | f(t) - f^k(t) |^2 \, dt \to 0, \qquad k \to \infty,$$

according as the coefficients, ϕ_n, of (8) do or do not satisfy the requirements of (I).

Corresponding to the cosine analogue, (I bis), of (I), there is an analogue of (II):

(II bis) (8) *can be replaced by* (11) *in* (II), *provided that* (10) *is satisfied* (*i. e., if* $\phi_0 = 0$).

First, if $c_1, c_2, \cdots$ is a fixed sequence of constants which need not satisfy (3), it follows from the completeness of the (L^2)-space that the existence of an $f(x)$ satisfying (18) is equivalent to

$$\int_0^{\pi} | f^j(t) - f^k(t) |^2 \, dt \to 0, \qquad j, k \to \infty.$$

In view of (16), this can be written in the form

$$\sum_{n=k+1}^{j} \sum_{m=k+1}^{j} c_n c_m^* \int_0^{\pi} \phi(nt) \phi^*(mt) \, dt \to 0, \qquad j, k \to \infty,$$

where $k \leqq j$. But the expression on the left of this limit relation is identical with the (k, j)-th remainder term of the *restricted* double series mentioned in (α), if $A = A^*$ in (α) is identified with the matrix (9). Hence, (II) follows from (α) and (I).

By using (ii) instead of (i), it will now be easy to deduce the following solution of the Töpler-Boltzmann problem in the (L^2)-case:

(III) *Let* $\phi_1, \phi_2, \cdots$ *be a sequence of constants having the property that the Dirichlet series* (5) *converges in the half-plane* $\sigma > 0$ *and satisfies both* (6) *and* (7). *Then* (8) *defines a function which is of class* (L^2) *and has the property that the relation*

$$f(t) \sim \sum_{n=1}^{\infty} c_n \phi(nt), \tag{19}$$

when interpreted as an expansion in the mean (L^2) [cf. (18), (16*)], *establishes a one-to-one correspondence between all sequences* $(c_1, c_2, \cdots)$ *satisfying* (3) *and all functions* $f(t)$ *of class* (L^2) *on* $(0, \pi)$.

The point is that the sequence

$$\phi(t), \phi(2t), \cdots, \phi(nt), \cdots \tag{20}$$

does not (in general) consist of orthogonal functions; so that the coefficients of (19) cannot be obtained in Fourier's fashion, and their characterization in terms of the minimum property, referred to in the first footnote, is not available.

What is claimed in (III) but not in (II) contains two corollaries neither of which is obvious; namely, that

(III′) *under the assumptions of* (III), *the sequence* (20) *is a basis of the functions* $f(t)$ *of class* (L^2) *on* $(0, \pi)$,

and that

(III″) *under the assumptions of* (III),

$$0 \sim \sum_{n=1}^{\infty} c_n \phi_n(nt) \text{ only when } \sum_{n=1}^{\infty} |c_n|^2 = 0 \qquad \left(\text{if } \sum_{n=1}^{\infty} |c_n|^2 < \infty\right); \tag{21}$$

in particular, the functions (20) *are linearly independent.*

(The latter statement is weaker than (21), since all that it claims is that, *when* k *is finite,*

$$\sum_{n=1}^{k} |c_n|^2 = 0, \text{ if } \sum_{n=1}^{k} c_n \phi(nt) = 0 \tag{21*}$$

holds almost everywhere.)

The basis assertion of (III′) means that, if an $f(t)$ of class (L^2) and an $\epsilon > 0$ are given, there exist a k and k constants C_n satisfying

$$\int_0^{\pi} \left| f(t) - \sum_{n=1}^{k} C_n \phi(nt) \right|^2 dt < \epsilon.$$

This is much less than (19), since the minimum property of the coefficients of (19) is not available. It should be noted in this regard that the assertion of the last formula line has been proved in [5], p. 566 and pp. 572-573, for the case

$$\phi(t) \sim \sum_{n=1}^{\infty} n^{-\lambda} \sin nt = \phi(t)$$

if $\lambda > \frac{1}{2}$, whereas the assertions of (III) for (19) are satisfied only if $\lambda > 1$.

The proof of (III) will involve (II). If (II bis) is used instead of (II), what results is the following variant of (III):

(III bis) (8) *can be replaced by* (11) *in* (III), *provided that* (19) *is replaced by*

(19 bis) $$f(t) \sim c_0 + \sum_{n=1}^{\infty} c_n \phi(nt),$$

and $(c_1, c_2, \cdots)$ *by* $(c_0, c_1, c_2, \cdots)$. *In particular,* (20) *in* (III′) *can then be replaced by*

(20 bis) $$1, \phi(t), \phi(2t), \cdots, \phi(nt), \cdots.$$

According to (i) and (ii), the assumptions of (III) mean that $D(\Phi)$ is a bounded matrix and has a (unique) bounded reciprocal matrix $D^{-1}(\Phi)$. But the boundedness of a matrix means that the matrix transforms every point of Hilbert's space into a point of Hilbert's space (Toeplitz). Hence, if $(f_1, f_2, \cdots)$ is any vector satisfying

(22) $$\sum_{n=1}^{\infty} |f_n|^2 < \infty,$$

it is transformed by $D^{-1}(\Phi)$ into a vector,

(23) $$(c_1, c_2, \cdots) = D^{-1}(\Phi)(f_1, f_2, \cdots),$$

satisfying (3). Furthermore, this mapping of the Hilbert space (f) on the Hilbert space (c) has a unique inverse on the latter space, since (23) has a (unique) bounded inverse,

(24) $$(f_1, f_2, \cdots) = D(\Phi)(c_1, c_2, \cdots).$$

Since $\sin t, \sin 2t, \cdots$ is a complete orthogonal system on $(0, \pi)$, the Fischer-Riesz assignment,

(25) $$f(t) \sim \sum_{n=1}^{\infty} f_n \sin nt,$$

establishes a one-to-one correspondence between the sequences $(f_1, f_2, \cdots)$ satisfying (22) and the functions $f(t)$ of class (L^2) on $(0, \pi)$. Consequently, (24) and (2) imply that the n-th Fourier sine constant, f_n, of every function, $f(t)$, of class (L^2) on $(0, \pi)$ can be represented in the form

$$f_n = \sum_{d|n} c_d \phi_{n/d}, \tag{26}$$

where $c_1, c_2, \cdots$ is a sequence satisfying (3). On the other hand, (II) states that any such sequence defines a function to which the partial sums, (16), of (17) tend in the mean (L^2) on $(0, \pi)$. Let $g(t) = g(t; c_1, c_2, \cdots)$ denote this limit function of class (L^2). Then, since

$$\int_0^\pi | g(t) - f^k(t) |^2 \, dt \to 0 \text{ as } k \to \infty,$$

holds for the functions (16) and since, if $g(t) \sim \sum_{n=1}^{\infty} g_n \sin nt$,

$$\int_0^\pi | g(t) - \sum_{m=1}^{j} g_m \sin mt |^2 \, dt \to 0 \text{ as } j \to \infty,$$

it follows from (8) that

$$g_n = \sum_{d|n} c_d \phi_{n/d}. \tag{27}$$

In fact, both of the limit processes, which lead to this evaluation of

$$\int_0^\pi g(t) \sin nt \, dt$$

or g_n, can be carried out "term-by-term," the legitimacy of the term-by-term integrations being assured by convergence in the mean (L^2).

According to (26) and (27), the two functions, $f(t)$ and $g(t)$, have the same Fourier sine constants, f_n and g_n, on $(0, \pi)$. It follows therefore from the uniqueness theorem of Fourier sine series (L^2) that $f(t) = g(t)$ (almost everywhere). If this is compared with the one-to-one correspondence established by (23) and (24) between Hilbert spaces (22) and (3), it is seen that, in order to complete the proof of (III), only the assertion, (21), of (III″) remains to be verified.

The proof of (21) can be based on the fact used after (14), namely, on the circumstance that the Hermitian matrix (9) is the product $D^*(\Phi)D(\Phi)$. This implies that the matrix (9) is non-negative definite. Actually, since $D(\Phi)$ has a bounded inverse, $D^{-1}(\Phi)$, the bounded matrix (9) must be positive

definite, that is to say such that $\mu \geqq 0$ can be refined to $\mu > 0$, if μ denotes the greatest lower bound attained by the Hermitian form belonging to the matrix (9), on the boundary of the unit sphere in Hilbert's space (Toeplitz). In particular,

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_n c_m^* \int_0^{\pi} \phi(nt)\phi^*(mt)\,dt = 0$$

holds only when $(c_1, c_2, \cdots) = (0, 0, \cdots)$. Since this double sum is the limit, as $k \to \infty$, of

$$\sum_{n=1}^{k} \sum_{m=1}^{k} c_n c_m^* \int_0^{\pi} \phi(nt)\phi^*(mt)\,dt = \int_0^{\pi} \left| 0 - \sum_{n=1}^{k} c_n \phi(nt) \right|^2 dt,$$

the assertion, (21), of (III″) follows.

This completes the proof of (III). The proof of (III bis) proceeds in the same way, if the preceding deduction is first applied to $f(t) - c_0$, that is, to the function which results if the constant term of the expansion (19 bis) is subtracted from the function $f(t)$.

In order to illustrate the nature of the limitations imposed on $\phi(t)$, it is instructive to consider the case in which the last assumption, (7), of (III) is relaxed to

(28) $$\Phi(s) \neq 0, \text{ where } \sigma > 0,$$

but all the other assumptions of (III) are retained. It is clear from (ii), and from the proof of (III), that the assertions of (III) cannot then remain true to their full extent. What is interesting is that not even (III′), a statement much weaker than (III), is such as to allow the reduction of (7) to (28).

In order to see this, it is sufficient to choose (8) as follows: $\phi(t) = \sin 2t$. Then every ϕ_n except $\phi_2 = 1$ is 0. Hence, (5) becomes $\Phi(s) = 2^{-s}$, and so (6) and the relaxed form, (28), of (7) are satisfied. But (7) is violated (in fact $|1/\Phi(s)| = 2^{\sigma} \to \infty$ as $\sigma \to \infty$), and this alone vitiates the assertion of (III′). For, since $\phi(t) = \sin 2t$, the sequence (20) now becomes

(28 bis) $$\sin 2t, \sin 4t, \sin 6t, \cdots,$$

a sequence which cannot be (L^2)-complete on $(0, \pi)$. In fact, this sequence is a (proper) subsequence of the sequence of all odd harmonics. But no such subsequence can be (L^2)-complete on $(0, \pi)$, since the sequence of all odd harmonics is an orthogonal sequence on $(0, \pi)$.

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ON RIEMANN'S REDUCTION OF DIRICHLET SERIES TO POWER SERIES.*

By Aurel Wintner.

Subject to provisos of convergence, the reduction in question connects

$$f(s)=\sum_{n=1}^{\infty}a_n/n^s \text{ and } F(r)=\sum_{n=1}^{\infty}a_nr^n$$

by the formal identity

$$\Gamma(s)f(s)=\int_0^{\infty}x^{s-1}F(e^{-x})dx$$

(cf. [8] and, for references concerning proofs of convergence in case of general Dirichlet series, [4], p. 11).

The Tauberian aspects of this connection seem to have been neglected in recent literature. An exception is one of the Hardy-Littlewood proofs of the prime number theorem ([3], pp. 128-133).

Today, it is possible to develop a Tauberian theory of the transformation $F\rightarrow f$ in a manner leading to more or less complete results. Curiously enough, even results containing $F(r)$ alone (cf. (III) below) can be obtained from the representation of f as the Mellin transform of F.

1. First, a few primitive facts will have to be collected.

(i) *If a Dirichlet series*

$$f(s)=\sum_{n=1}^{\infty}a_n/n^s \tag{1}$$

is convergent in the half-plane $\sigma>\lambda$, then the corresponding power series

$$F(r)=\sum_{n=1}^{\infty}a_nr^n \tag{2}$$

is convergent when $r<1$ and satisfies

$$F(r)=O(1-r)^{-c} \text{ if } c>\max(0,\lambda), \tag{3}$$

as $r\rightarrow 1$.

* Received February 14, 1947.

In fact, if $\lambda \geqq 0$, then the convergence of (1) for $\sigma > \lambda$ means that

$$\sum_{m=1}^{n} a_m = O(n^{\lambda+\epsilon}) \tag{4}$$

holds for every $\epsilon > 0$. This implies that (2) is convergent when $r < 1$ and that, since

$$\sum_{n=1}^{\infty} a_n r^n = (1-r) \sum_{n=1}^{\infty} \sum_{m=1}^{n} a_m r^n, \tag{5}$$

the function $F(r)$ is

$$(1-r) \sum_{n=1}^{\infty} O(n^{\lambda+\epsilon}) r^n = (1-r) O(1-r)^{-\lambda-\epsilon-1}$$

as $r \to 1$. This proves (3) under the assumption $\lambda \geqq 0$. But this assumption is made superfluous by the wording of (3).

(ii) *If the abscissa of convergence of* (1) *is non-negative and finite, then the integral*

$$\int_0^{\infty} x^{s-1} F(e^{-x})\, dx, \qquad \text{(cf. (2))}, \tag{6}$$

which is, when $\sigma < 1$, improper at $x = 0$, is absolutely convergent within the half-plane of (*not necessarily absolute*) *convergence of* (1).

The half-planes are meant to be open.

If λ denotes the abscissa of convergence of (1), the assumptions of (ii) are $0 \leqq \lambda < \infty$. According to (i), these imply that $F(r) = O(1-r)^{-\lambda-\epsilon}$ as $r \to 1-0$, where $\epsilon > 0$ is arbitrary. Since this means $F(e^{-x}) = O(x^{-\lambda-\epsilon})$ as $x \to +0$, it follows that the contribution of the range $0 < x \leqq 1$ to (6) is absolutely convergent if the real part, σ, of s satisfies the inequality

$$(\sigma - 1) - (\lambda + \epsilon) > -1,$$

that is, if $\sigma > \lambda$. On the other hand, the contribution of the range $1 \leqq x < \infty$ to (6) is absolutely convergent for every s, since

$$F(e^{-x}) = O(e^{-x}) \text{ as } x \to \infty, \tag{7}$$

by (2) (where $F(0) = 0$, hence $F(r) = O(r)$ as $r \to 0$). This proves (ii).

(iii) *If* (1) *is convergent when $\sigma > \lambda$, and if $\lambda \geqq 0$, then*

$$f(s) = \int_0^{\infty} x^{s-1} F(e^{-x})\, dx / \Gamma(s) \tag{8}$$

when $\sigma > \lambda$.

This is that connection between (1) and (2) on which Riemann's proofs of the functional equation of $\zeta(s)$ depend. His starting point is that (8), for $\sigma > 0$, is obvious from

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx, \qquad \sigma > 0, \tag{9}$$

if (1) and (2) reduce to monomials, $f(s) = 1/n^s$ and $F(r) = r^n$. Hence, (8) is true, for reasons of distributivity, in the half-plane $\sigma > 0$, if (1) is a finite sum. This reduces the problem to one concerning the legitimacy of a term-by-term integration in an integral (which is improper at $x = \infty$ and, if $\sigma < 1$, at $x = 0$ also). This term-by-term integration can be justified from (7) and (3), if $\sigma > \max(0, \lambda)$.

The last step in this proof of (iii) is just a routine matter. Nevertheless, this, the traditional, verification of (iii) is by no means as straightforward as it appears. In fact, it thinks of the problem as one concerning the *division of* (6) *by* (9), whereas the actual problem is that of the *multiplication of* (1) *by* (9). Correspondingly, a proof of (iii) which is less artificial than the usual one and which, without additional labor, leads much further than (iii), is contained in a general lemma concerning multiplication. In fact, (1) can be written as a *convergent,* and (9) as an *absolutely convergent,* Mellin integral, and so nothing more is involved than the integral form of the Mertens-Stieltjes multiplication theorem on series.

2. An immediate consequence of (iii) is the following fact, which claims that, if (1) is convergent for $\sigma > 1$ and if the coefficient sequence $a_1, a_2, \cdots$, has an "A-mean," then the sequence has a "D-mean" as well (and the latter is equal to the former):

(iv) *If the Dirichlet series* (1) *is convergent for* $\sigma > 1$ *and if the corresponding power series* (2) *(which, by* (i), *must then converge for* $r < 1$) *is such that*

$$(1 - r)F(r) \to l \text{ as } r \to 1 \tag{10}$$

holds for some $l =$ const., *then*

$$\epsilon f(1 + \epsilon) \to l \text{ as } \epsilon \to 0. \tag{11}$$

The implication claimed by (iv) is well-known. It can be completed by the following *necessary condition* for (10) (if (1) is convergent when $\sigma > 1$):

(v) *If t is fixed, then, under the assumptions of* (iv),

$$(12) \qquad \epsilon f(1+\epsilon+it) \to 0 \text{ as } \epsilon \to 0, \text{ if } 0 < \pm t < \infty$$

(*but* (12) *must be replaced by* (11) *at* $t = 0$).

From the proof of the necessity of $\zeta(1+it) \neq 0$ for the truth of the prime number theorem, standard is that particular case of (v) in which the (A)-assumption, (10), is strengthened to the corresponding $(C, 1)$-assumption,

$$(13) \qquad (a_1 + \cdots + a_n)/n \to l \qquad (n \to \infty).$$

Actually, both (iv) and (v) can be concluded from (iii).

In fact, the assumptions of (iii) are now satisfied if $\sigma > 1$. Hence, from (8),

$$(14) \qquad \Gamma(1+\epsilon+it)f(1+\epsilon+it) = \int_0^\infty x^{\epsilon+it}F(e^{-x})dx,$$

if t is real and $\epsilon > 0$. But the assumption (10) means that

$$(15) \qquad F(e^{-x}) = lx^{-1} + o(x^{-1}) \text{ as } x \to 0.$$

It follows therefore from (7) that, as $\epsilon \to 0$, the integral on the right of (14) is

$$\int_0^1 x^{\epsilon+it}lx^{-1}dx + O(1) = l\int_0^1 x^{-1+\epsilon+it}dx + O(1) = l/(\epsilon+it) + O(1).$$

Consequently, if (14) is multiplied by ϵ,

$$\epsilon\Gamma(1+\epsilon+it)f(1+\epsilon+it) = l\epsilon/(\epsilon+it) + O(\epsilon).$$

This relation proves the assertions, (11) and (12), of (iv) and (v), since $\Gamma(1+\epsilon) \to 1$ if $t = 0$ and $\Gamma(1+\epsilon+it) \to \Gamma(1+it) \neq 0$ if $t \neq 0$.

3. According to an "elementary" theorem of Hardy and Littlewood, proved very simply by Karamata [6], the Tauberian restriction

$$(16) \qquad a_n \geqq 0$$

is sufficient in order that the converse of Frobenius' unconditional implication, (13) $\to$ (10), be true. In other words,

(17) (10) and (16) imply (13).

On the other hand,

(11) and (16) do not imply (13)

or, what (in view of (17)) is the same thing,

(18) (11) and (16) do not imply (10).

In fact, (18) is clear from Dedekind's example,

(19) $a_n = n$ if $n = 2^k$ and $a_n = 0$ if $n \neq 2^k$, $(k = 0, 1, \cdots)$,

(cf. [1]). What is responsible for this situation is precisely (v). For, in order to prove (18), it is sufficient to ascertain that the convergence of (1) for $\sigma > 1$ and the assumptions (11), (16) together do not imply the assertion, (12), of (v). But (16) is satisfied in the case (19), and the series (1) becomes

$$f(s) = \sum_{k=0}^{\infty} 2^k/2^{ks} = \sum_{k=1}^{\infty} (2^{-s+1})^k,$$

a geometric progression which converges when $\sigma > 1$. However, since $f(s)$ has the period $2\pi i/\log 2$ and possesses a simple pole at $s = 1$, the necessary condition expressed by (12) is violated at $t = \pm 2\pi i n/\log 2$, even though (11) is satisfied.

In order to satisfy (12), Ikehara's theorem, which retains (16) and the convergence of (1) for $\sigma > 1$, assumes both (11) and (12), and somewhat more; substantially, the existence of a continuous boundary function for

$$f(s) - l/(s-1)$$

on the line $\sigma = 1$ (cf., e.g., [7]).

Since the local behavior of $l/(s-1)$ on $\sigma = 1$ is the same as that of $l\zeta(s)$, Ikehara's theorem can be formulated as follows:

If the Dirichlet series

(20) $$f^*(s) = f(s) - l\zeta(s) = \sum_{n=1}^{\infty}(a_n - l)/n^s$$

converges in the half-plane $\sigma > 1$ to a function which attains continuous boundary values on the line $\sigma = 1$, then (16) is sufficient for (13).

4. This theorem will now be refined so as to replace Ikehara's assertion, (13), by (10) and Ikehara's Tauberian restriction, (16), by a more inclusive assumption, to be placed on the *function* (2) (which represents *averages* of $a_1, a_2, \cdots$), rather than on the *individual coefficients* a_n, as follows:

(vi) *Let the Dirichlet series* (1) *be convergent in the half-plane* $\sigma > 1$.

Then the Abelian relation (10) *holds for the power series* (2) *whenever the coefficients* a_n *satisfy the following pair of conditions*:

(α) $F(r)=\sum_{n=1}^{\infty} a_n r^n$ *is a monotone function of* r *as* $r\rightarrow 1$ (*i. e., ultimately*), *and*

(β) *the function* (20), *where* $\sigma>1$, *goes over into a continuous boundary function on the line* $\sigma=1$; *a requirement which can be relaxed to the assumption that*

(β bis) *for every positive* $T<\infty$, *the functions*

$$f_\sigma^*(t)=f^*(\sigma+it), \qquad -\infty<t<\infty, \tag{21}$$

defined by (20) *for* $\sigma>1$, *converge, as* $\sigma\rightarrow 1$, *in the mean of the function space* $(L)=(L^1)$ *on* $(-T,T)$ (*that is*,

$$\int_{-T}^{T} |f^*(1+\delta+it)-f^*(1+\eta+it)|\,dt\rightarrow 0 \text{ as } \delta^2+\eta^2\rightarrow 0, \tag{22}$$

where $\delta>0$ *is independent of* $\eta>0$).

Remark. In order that (22) be satisfied, it is sufficient, but not necessary, that the functions (21) of t be majorized by an L-integrable function of t on $(-T,T)$, as $\sigma\rightarrow 1$.

Since (16) is sufficient for (α) in (vi), it follows from (17) that Ikehara's theorem is contained in (vi). The converse inference is not possible in any sense. In fact, a domain which is reached by (vi) but cannot be reached by Ikehara's theorem is exemplified by so primitive a case as

$$a_n=(-1)^{n+1}n. \tag{23}$$

For, since $a_n\neq o(n)$ in this case, the assertion, (13), of Ikehara's theorem is surely false. But the assertion, (10), of (vi) is true; and the point is that this can be concluded from (vi) itself. In fact, (1) converges, in the case (23), in the half-plane $\sigma>1$ to a function, $f(s)$, which is an entire function, hence such as to satisfy (β) (with $l=0$), and (α) is fulfilled, since (2) becomes

$$F(r)=\sum_{n=1}^{\infty}(-1)^{n+1}nr^n=r/(1+r)^2\rightarrow \tfrac{1}{4} \qquad (1>r\rightarrow 1).$$

5. In order to prove (vi), suppose first only that (1) is convergent

when $\sigma > 1$. This means that the assumption of (iii) is satisfied by $\lambda = 1$. Hence, (8) is applicable if the real part of $s = \sigma + it$ exceeds 1. Accordingly,

$$f(s)\Gamma(s) = \int_{-\infty}^{\infty} e^{-sy} F(\exp - e^{-y})\,dy, \tag{25}$$

if x is replaced by e^{-y} in (8).

In view of (i), the integral (25) is absolutely convergent. Hence, if $\phi(t)$ is any L-integrable function on a finite interval $(-T, T)$, then

$$\int_{-T}^{T}\int_{-\infty}^{\infty} | \phi(t) e^{-sy} F(\exp - e^{-y}) | \,dy\,dt < \infty. \tag{26}$$

Consequently, from (25), and by Fubini's theorem,

$$\int_{-2T}^{2T} \phi(t) f(s)\Gamma(s)\,dt = \int_{-\infty}^{\infty} F(\exp - e^{-y}) e^{-\sigma y} \int_{-2T}^{2T} e^{-ity}\phi(t)\,dt\,dy, \tag{27}$$

if T is replaced by $2T$.

As in the standard proof of Ikehara's theorem (cf., e. g., [7]), choose

$$\phi(t) = \tfrac{1}{2}(1 - \tfrac{1}{2} | t | /T) e^{ixt},$$

where x is a real number. Then the interior integral on the right of (27) becomes

$$\tfrac{1}{2}\int_{-2T}^{2T} e^{i(x-y)t}(1 - \tfrac{1}{2} | t | /T)\,dt = T\Phi(Tx - Ty),$$

if Φ is an abbreviation for the function

$$\Phi(u) = (\sin u)^2/u^2. \tag{28}$$

Hence, (27) reduces to

$$\begin{aligned} &\tfrac{1}{2}\int_{-2T}^{2T} (1 - \tfrac{1}{2} | t | /T) e^{ixt} f(\sigma + it)\Gamma(\sigma + it)\,dt \\ &= T\int_{-\infty}^{\infty} F(\exp - e^{-y}) e^{-\sigma y}\Phi(Tx - Ty)\,dy, \end{aligned} \tag{29}$$

where $\sigma > 1$.

Clearly, (29) remains true if f and F are replaced by f^* and F^* respectively, where, corresponding to (1), (2) and (20),

$$(30) \qquad F^*(r) = \sum_{n=1}^{\infty} (a_n - l)r^n = F(r) - lr/(1-r).$$

Let (31) denote the identity which results from (29) upon the replacement $(f, F) \to (f^*, F^*)$.

In view of the completeness of the L-space, the last assumption, (β bis), of (vi) implies that, as $\sigma \to 1$, the functions (21) tend in the mean (L) to an L-integrable function, say $f_0(t)$, on every finite t-interval. Furthermore, this limit process can be carried out beneath the integral sign of the integral on the left of (31) (the convergence being "strong" with reference to the L-space). Finally, since (31) is true when $\sigma > 1$, the expression on the right of (31) must tend, as $\sigma \to 1$, to a finite limit, the latter being the limit of the expression on the left. Accordingly, if

$$(32) \qquad g_T(t) = \tfrac{1}{2}(1 - \tfrac{1}{2}\,|\,t\,|\,/T)f_0(t)\Gamma(1+it)$$

and

$$(33) \qquad \phi_T(x) = T \lim_{\sigma \to 1} \int_{-\infty}^{\infty} F^*(\exp - e^{-y})e^{-\sigma y}\Phi(Tx - Ty)\,dy,$$

the result of the limit process in (31) is

$$\int_{-2T}^{2T} e^{ixt} g_T(t)\,dt = \phi_T(x),$$

along with the existence of (33) (as a finite limit) for every real x and for every $T > 0$. Finally, since $f_0(t)$, and therefore the function (32) of t, is L-integrable on every finite t-interval, the last formula line shows that, in view of the Riemann-Lebesgue lemma,

$$(34) \qquad \phi_T(x) \to 0 \text{ as } x \to \pm\infty.$$

6. Condition (α) of (vi) has not been used thus far. Hence (if f^*, F^* are replaced by f, F), the result of the preceding deduction can be formulated as follows:

(vii) *If the Dirichlet series* (1) *is convergent when* $\sigma > 1$ *and satisfies, for a fixed* $T > 0$, *the condition*

$$\int_{-2T}^{2T} |f(1+\delta+it)-f(1+\eta+it)|\,dt \to 0 \text{ as } (\delta,\eta)\to(0,0), \tag{35}$$

then the Abelian value,

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} e^{-\epsilon y}F(\exp - e^{-y})e^{-y}\Phi(Tx-Ty)\,dy, \tag{36}$$

of the improper integral

$$\int_{-\infty}^{\infty} F(\exp - e^{-y})e^{-y}\Phi(Tx-Ty)\,dy, \tag{37}$$

where F, Φ *are defined by* (2), (28), *exists, as a finite limit, for every real* x, *and*

$$(36) \text{ tends to } 0 \text{ as } x\to\pm\infty. \tag{38}$$

The convergence of (37) is not claimed. The transition from (36) to (37) would correspond to one from the (A)-summability of a series Σc_n to its convergence, that is, from

$$\sum_{n=1}^{\infty} c_n e^{-\epsilon n} \to C \text{ as } \epsilon\to 0 \tag{36$'$}$$

to

$$\sum_{n=1}^{N} c_n \to C \text{ as } N\to\infty, \tag{37$'$}$$

which, according to Tauber's theorem, is justified if (and only if)

$$\sum_{n=1}^{N} nc_n = o(N). \tag{39$'$}$$

Correspondingly, by the integral analogue of Tauber's theorem,

(vii bis) (36) *in* (vii) *can be replaced by* (37) *if* (*and only if*)

$$\int_{0}^{R} yF(\exp - e^{-y})e^{-y}\Phi(Tx-Ty)\,dy = o(R) \text{ as } R\to\infty, \tag{39}$$

where x *is fixed* (*as is* T).

Strictly speaking, (39) belongs to those modifications of (36), (37) in which the lower limit of integration, instead of being $y = -\infty$, is $y = 0$, as in (39). Actually, the contribution of the half-line $y < 0$ to (36), (37)

is immaterial, since, if $r = \exp - e^{-y}$, this half-line represents an r-interval ending *before* the crucial point, $r = 1$, in (2).

A condition more superficial than (39) is

$$F(r) \geqq 0 \tag{40}$$

(as $r \to 1$). In fact, by (28),

$$\Phi \geqq 0, \tag{41}$$

and so it is clear that (36) can be replaced by (37) if (but not only if) (40) is satisfied.

7. After this interruption of the proof of (vi), consider again (20), (30), instead of (1), (2), hence (33), instead of (36).

If the analogue of Tauber's condition, (39′), can be assumed, then (33) simplifies to

$$\phi_T(x) = T \int_{-\infty}^{\infty} F^*(\exp - e^{-y}) e^{-y} \Phi(Tx - Ty)\, dy, \tag{42}$$

and this would admit a direct application of (α) along the lines of the "Tauberian" argument of Karamata-Wiener. Fortunately, the circumstance that the integral (42) must be confined to its Abelian form, (33), will not lead to difficulties.

First, the asumption (α) is that the function (2) is monotone when r is close enough to its limiting value, 1. After a suitable alteration of the first coefficient, a_1, of (2), an alteration which clearly has no influence on the assertion and the remaining assumptions of (vi), it can be assumed that $F(r)$ is monotone from the beginning, that is, on the whole interval $0 < r < 1$. Hence, $F(r)$ will be strictly monotone, and will not therefore become 0 more than once, when r varies from $r = 0$ to $r = 1$. After an additional alteration of a_1, it can be assumed that $F(r)$ does not become 0 at all (except at $r = 0$), and so, since $F(r)$ can be replaced by $-F(r)$, that

$$F(r) > 0 \text{ if } 0 < r < 1. \tag{43}$$

Two cases are possible, according as the monotone function $F(r)$ is decreasing or increasing. In the first case, (43) implies that $F(r)$ is bounded as $r \to 1$ and must therefore tend to a finite limit, $F(1-0)$. Since this implies what is claimed in (vi) (and much more), it is sufficient to consider the second case, where

$$F(r') < F(r'') \text{ if } r' < r''. \tag{44}$$

It should be noted that, in the above normalization, the formal "residue" occurring in (20), (30) cannot be negative, since (2) and (43) imply that

$$l \geqq 0. \tag{45}$$

Incidentally, the proof of (vi) admits of a slight simplification if the sign of equality holds in (45), since (30) then reduces to (2). But the assertion of (vi) is not more evident in this limiting case than it is when $l \neq 0$. Actually, the contribution of the trivial second term in (30) or, if

$$F_0(r) = \sum_{n=1}^{\infty} r^n = r/(1-r), \tag{46}$$

in

$$F^*(r) = F(r) - lF_0(r), \tag{47}$$

can be isolated in any case.

8. According to (47), the contribution of (46) to the integral following the lim-sign in (33) is

$$\int_{-\infty}^{\infty} E_\sigma(y)\Phi(Tx - Ty)\,dy,$$

where

$$E_\sigma(y) = \exp(-e^{-y} - \sigma y)/(1 - \exp - e^{-y})$$

and $\sigma > 1$. If $\sigma \to 1$, this quotient tends to

$$E(y) = \exp\,(-e^{-y} - y)/(1 - \exp - e^{-y}), \tag{48}$$

and therefore the last integral to

$$\int_{-\infty}^{\infty} E(y)\Phi(Tx - Ty)\,dy. \tag{49}$$

In fact, it is clear from (48) that

$$E(y) > 0 \text{ for } -\infty < y < \infty$$

and, as $y \to \infty$,

$$E(y) \sim \exp\,(-e^{-y} - y)/e^{-y} = \exp\,(-e^{-y}) \to 1, \qquad (y \to \infty),$$

finally, as $y \to -\infty$,

$$E(y) = o(1)/(1 - o(1)) = o(1), \qquad (y \to -\infty).$$

On the other hand, from (28),

$$\Phi(Tx - Ty) = O(y^{-2}) \text{ as } y \to \pm\infty,$$

if x and T are fixed. But the four last formula lines imply the convergence of the integral (49), and the legitimacy of the limit process $(\sigma \rightarrow 1)$ which led to (49) is just as obvious.

Accordingly, the contribution of the second term on the right of (47) to the limit on the right of (33) is $-l$ times the value of the (convergent) integral (49). Hence, (33) can be decomposed into

$$(50) \qquad \phi_T(x) = T \lim_{\sigma \rightarrow 1} \int_{-\infty}^{\infty} e^{-\sigma y} F(\exp -e^{-y}) \Phi(Tx - Ty)\, dy - lT\psi_T(x),$$

where, according to (49) and (48),

$$(51) \qquad \psi_T(x) = \int_{-\infty}^{\infty} e^{-Y-y}(1 - e^{-Y})^{-1} \Phi(Tx - Ty)\, dy, \qquad Y = e^{-y}.$$

Since the limit occurring on the right of (50) exists, it follows from (43) and (41) that (50) can be replaced by

$$(52) \qquad \phi_T(x) = T \int_{-\infty}^{\infty} e^{-y} F(\exp -e^{-y}) \Phi(Tx - Ty)\, dy - lT\psi_T(x).$$

In fact, this reduction of (50) depends only on the trivial criterion, (40), mentioned after the necessary and sufficient condition, (vii bis).

Needless to say, what this reduction actually accomplishes is that the representation (33) can be improved to (42).

9. It is seen from (51), (28), and from the four formula lines preceding (50), that

$$\lim_{x \rightarrow \infty} \psi_T(x) = \int_{-\infty}^{\infty} \Phi(Ty)\, dy.$$

The last integral is

$$\int_{-\infty}^{\infty} \Phi(Ty)\, dy = \int_{-\infty}^{\infty} \Phi(y)\, dy/T = \pi/T,$$

by (28). According to (34),

$$\lim_{x \rightarrow \infty} \phi_T(x) = 0.$$

Finally, (52) can be written in the form

$$\phi_T(x) + lT\psi_T(x) = \mu_T(x),$$

where

$$\mu_T(x) = \int_{-\infty}^{\infty} e^{-y/T} F(\exp - e^{-y/T}) \Phi(Tx - y)\, dy. \tag{53}$$

The four formula lines which precede (53) imply that

$$\lim_{x\to\infty} \mu_T(x) = \pi l, \text{ where } \pi = \int_{-\infty}^{\infty} \Phi(y)\, dy, \tag{54}$$

is an identity in T. On the other hand, the assertion of (vi) is (10) or, what is the same thing,

$$\lim_{u\to\infty} e^{-u} F(\exp - e^{-u}) = l. \tag{55}$$

Hence, the proof of (vi) is complete if it is verified that (53), (54) and (41) imply (55) by virtue of *both* assumptions (43), (44), the *second* of which has not been used thus far. But the proof of this implication is exactly the same as in the corresponding final step in the proof of Ikehara's theorem (namely, steps 1)-2) in [7], pp. 526-527) and will therefore be omitted.

10. The theorem just proved, (vi), implies the following:

(viii) *If a Dirichlet series* (1) *is convergent for* $\sigma > 1$ *and, for some constant* l*, the corresponding function* (20) *satisfies the boundary condition* (22) *for every positive* $T < \infty$*, then*

$$(1 - r) \sum_{n=1}^{\infty} a_n r^n \to l \text{ as } r \to 1 \tag{56}$$

holds whenever the coefficients of (1) *satisfy the estimate*

$$\sum_{n=1}^{x} a_n = O(x). \tag{57}$$

Clearly, the restriction

$$a_n = O(1) \tag{58}$$

is sufficient for (57). Under this restriction, Ikehara's theorem supplies (13), instead of just (56). Nevertheless, it is seen from (17) that (13) can be concluded from (viii) also, if the Tauberian conditions of (viii) are strengthened to those of Ikehara's theorem. Conversely, if the latter restrictions are replaced by those of (viii), then (56) can hold when (13) is false. This is shown by the example (23), which satisfies (57) and (22) (with $l = 0$ in (20)).

The true theorem is, however, one which, instead of the partial sums

(59) $$s_n = \sum_{m=1}^{n} a_m$$

occurring in (viii), involves only the sums

(60) $$t_n = \sum_{m=1}^{n} m a_m,$$

which occur in Tauber's own theorem. In fact, from (59) and (60),

$$t_n = n s_n - \sum_{m=1}^{n-1} s_m.$$

Hence,

$$\text{if } s_n = O(n), \text{ then } t_n = O(n^2).$$

Consequently, (viii) is contained in the following theorem:

(ix) *In* (viii), *the coefficient restrictions can be replaced by*

(61) $$\sum_{n=1}^{x} n a_n = O_L(x^2)$$

(*in particular,* (16) *can be relaxed to its averaged form,*

(62) $$\sum_{n=1}^{x} n a_n \geqq 0,$$

and (57) *to*

(63) $$\sum_{n=1}^{x} n a_n = O(x^2),$$

the corresponding averaged condition).

It is readily verified that, by adding a constant to every a_n, one can reduce the sufficiency of (61) to that of (62). Hence, in order to prove both (viii) and (ix), it is sufficient to show that condition (α) of (vi) is satisfied if (62) is assumed. In other words, it is sufficient to ascertain that the derivative of (2) is non-negative if the sums t_n, defined by (60), are non-negative. But this is obvious, since the derivative of (2) is

$$\sum_{n=1}^{\infty} n a_n r^{n-1} = \sum_{n=1}^{\infty} (t_n - t_{n-1}) r^{n-1} = (1-r) \sum_{n=1}^{\infty} t_n r^{n-1}.$$

11. The rôle of (61) in (ix) is that of supplying a sufficient condition for the first assumption, (α), of (vi). It will now be shown that a sufficient condition for the remaining assumption, (β bis), of (vi) (that is, for (22), where $0 < T < \infty$), is supplied by

(64) $$\int_0^{1-0} (1-r)\,|\,F^*(r)\,|^2\,dr < \infty.$$

(x) *Let*

(65) $$f(s) = \sum_{n=1}^{\infty} a_n/n^s, \text{ hence } f^*(s) = \sum_{n=1}^{\infty} (a_n - l)n^s,$$

be convergent for $\sigma > 1$ *(and, therefore,*

(66) $$F(r) = \sum_{n=1}^{\infty} a_n r^n \text{ and } F^*(r) = \sum_{n=1}^{\infty} (a_n - l) r^n$$

for $r < 1$*). Then* (61) *and* (64)*, respectively, are sufficient for the assumptions* (α) *and* (β bis) *of* (vi) *and imply, therefore, that*

(67) $$(1-r)F(r) \to l, \text{ i.e., } F^*(r) = o(1-r)^{-1}, \text{ as } r \to 1.$$

There is something unsatisfactory in the structure of the last conclusion, since both the assumptions, (61) and (64), and the assertion, (67), involve only the power series, (66). Thus the sole part played by the Dirichlet series, (65), consists in their convergence in the half-plane $\sigma > 1$. This, an assumption *not* implied by the convergence of the power series for $r < 1$, enters, of course, via (vi).

This anomaly can be disposed of, since (vi) can be generalized as follows:

(I) *Let* $F(r) = \sum_{n=1}^{\infty} a_n r^n$ *and*

(68) $$\phi(s) = \int_0^{\infty} x^{s-1} F(e^{-x})\,dx$$

be convergent for $r < 1$ *and* $\sigma > 1$ *respectively. Suppose that*

(69) $$F(r) \text{ is monotone as } r \to 1-0$$

(for instance, that (58) *is satisfied) and that the function*

(70) $$\phi^*(s) = \phi(s) - l\zeta(s)$$

or, equivalently, the function

(71) $$\phi^*(s) = \phi(s) - l/(s-1),$$

where l *is a constant, satisfies the boundary condition*

$$(72)\qquad \int_{-T}^{T} | \phi^*(1+\delta+it) - \phi^*(1+\eta+it) | \, dt \to 0 \text{ as } (\delta,\eta) \to (0,0),$$

for every fixed $T < \infty$. Then, as $r \to 1$,

$$(73)\qquad (1-r)F(r) \to l, \text{ i. e., } \sum_{n=1}^{\infty} (a_n - l) r^n = o(1-r)^{-1}.$$

The point is that this theorem, (I), includes Dirichlet series (65) which have no half-planes of convergence. Nevertheless, the proof of (I) is the same as that of (vi). In fact, if $f^*(s)$, instead of being defined by (20), is defined by

$$(74)\qquad f^*(s) = \phi^*(s)/\Gamma(s), \qquad (\sigma > 1),$$

where $\phi^*(s)$ is the function (70), then, since $\Gamma(s)$ is continuous and distinct from 0 on the closed half-plane $\sigma \geqq 1$, the assumption (72) of (I) becomes equivalent to the assumption (β bis) of (vi), whereas (69) represents (α) in (vi). But this suffices for the proof of (vi). For, on the other hand, (iii) implies that, *if* the Dirichlet series (1) or (20) is convergent for $\sigma > 1$, then (74) holds by virtue of (68) and (70), and, on the other hand, only the functicns (68), (70), (74), rather than the two series (65), have been used in the proof of (vi).

12. This reduces the problem of (x) to one involving the power series (66) only. In fact, (x) can be reduced to (I) and to the following criterion:

(II) *If a power series*

$$(75)\qquad G(r) = \sum_{n=1}^{\infty} b_n r^n$$

and the corresponding Mellin transform

$$(76)\qquad \psi(s) = \int_{0}^{\infty} x^{s-1} G(e^{-x}) \, dx$$

are convergent for $r < 1$ and $\sigma > 1$ respectively, and if

$$(77)\qquad \int_{0}^{1-0} (1-r) | G(r) |^2 \, dr < \infty,$$

then the function $\psi(s) = \psi(\sigma + it)$, where $\sigma > 1$, goes over into a locally

L-integrable boundary function on the line $\sigma = 1$ (that is, there exists a measurable function $\psi_0(t)$, $-\infty < t < \infty$, satisfying

$$(78) \qquad \int_{-T}^{T} |\, \psi(1 + \epsilon + it) - \psi_0(t) \,|\, dt \to 0 \text{ as } \epsilon \to 0,$$

where $T < \infty$ is arbitrary).

Actually, the convergence of (76) in the half-plane $\sigma > 1$ need not be required, since even the *absolute* convergence of (76), for $\sigma > 1$, is implied by (77). In fact, (76) is convergent in the half-plane $\sigma > 1$ if and only if the improper integral

$$\int^{1-0} (\log r)^{\epsilon}\, G(r)\, dr/r$$

is convergent for every $\epsilon > 0$. Hence, (76) is absolutely convergent in the half-plane $\sigma > 1$ if and only if

$$\int_0^1 |(1 - r)^{\epsilon}\, G(r)\,|\, dr < \infty \quad \text{when} \quad \epsilon > 0.$$

But Schwarz's inequality shows that the last integral is majorized by the square root of JC_ϵ, where J denotes the integral (77) and

$$C_\epsilon = \int_0^1 |(1 - r)^{-\frac{1}{2}+\epsilon}\,|^2\, dr = \int_0^1 (1 - r)^{-1+2\epsilon}\, dr < \infty.$$

Clearly, (78) implies that

$$(79) \qquad \int_{-T}^{T} |\, \psi(1 + \delta + it) - \psi(1 + \eta + it) \,|\, dt \to 0 \text{ as } (\delta, \eta) \to (0, 0),$$

i.e., that (72) is satisfied for $\phi^* = \psi$. Hence, if (II) is granted, (I) supplies the following Tauberian theorem, which involves nothing but a power series:

(III) *If $r < 1$, let*

$$(80) \qquad P(r) = \sum_{n=1}^{\infty} c_n r^n$$

be a convergent power series satisfying the integral condition

(81) $$\int_0^{1-0} (1-r)\,|\,P(r)\,|^2\, dr < \infty,$$

and suppose that, as $r \to 1$,

(82) $P(r)$ is a monotone function

(*which is sure to be the case if*

(82*) $$\sum_{n=1}^{x} nc_n = O_L(x^2) \text{ as } x \to \infty;$$

for instance, if

(82′) $$\sum_{n=1}^{x} nc_n \geqq 0$$

holds from a certain x *onward*). *Then*

(83) $$P(r) = o(1-r)^{-1} \text{ as } r \to 1.$$

Clearly, (x) is a corollary of (III). Since (I) and (II) imply (III), and since (I) has already been verified, it follows that only (II) remains to be proved.

13. The assertion of (II) is the existence of a measurable function $\psi_0(t)$, $-\infty < t < \infty$, satisfying (78), where T is arbitrary ($< \infty$). In view of the completeness of the L-space on a t-interval, the existence of such a $\psi_0(t)$ is equivalent to (79). On the other hand, since $T < \infty$, it follows from Schwarz's inequality that (79) is sure to be true if $(79^2)_T$ is true, where $(79^2)_T$ denotes the relation which results if the function integrated in (79) is replaced by its square. But $(79^2)_T$, where $T < \infty$, is certainly true if $(79^2)_\infty$ is true (with the understanding that $\psi(\sigma + it)$ should be of class (L^2) on the line $-\infty < t < \infty$, if $\sigma > 1$ is fixed). Finally, Plancherel's theorem states that $(79^2)_\infty$ is true if and only if

(84) $$\int_{-\infty}^{\infty} |\,\Psi_\delta(u) - \Psi_\eta(u)\,|^2\, du \to 0 \text{ as } (\delta, \eta) \to (0, 0),$$

where (in the (L^2)-sense)

(85) $$2\pi\Psi_\epsilon(u) \sim \int_{-\infty}^{\infty} \psi(1 + \epsilon + it) e^{itu} dt$$

and, for every $\epsilon > 0$,

$$(86) \qquad \int_{-\infty}^{\infty} |\psi(1+\epsilon+it)|^2 \, dt < \infty, \text{ i.e., } \int_{-\infty}^{\infty} |\Psi_\epsilon(u)|^2 \, du < \infty.$$

Consequently, more than (II) will be proved if it is verified that the assumptions of (II) imply (86) and (84), where Ψ_ϵ is defined by (85) in virtue of (86).

To this end, put $s = 1+\epsilon+it$ and $x = e^{-u}$ in (76). Then (76) appears in the form

$$\psi(1+\epsilon+it) = \int_{-\infty}^{\infty} e^{-(1+\epsilon)u} G(\exp - e^{-u}) e^{-itu} du.$$

As verified after (II), this integral is absolutely convergent by virtue of the assumption (77) of (II). Since (77) implies that the function

$$e^{-(1+\epsilon)u} \; G(\exp - e^{-u})$$

is of class (L^2) (cf. below) and since, being analytic, it is of bounded variation locally, Fourier's inversion is legitimate. This means that the integral on the right of (85) is convergent (beforehand, as a "principal" integral in Cauchy's sense), and is equal to

$$2\pi e^{-(1+\epsilon)u} \; G(\exp - e^{-u}).$$

But now it is clear that (in view of the absolute convergence of (75) and (76) for $r < 1$ and $\sigma > 1$, respectively) the condition (86) is satisfied, and that the function on the left of (85) is precisely the function in the last formula line; in fact, the Fourier transform (L^2) of a function of class (L^2) is (L^2)-unique.

It also follows that (85), the only relation which remains to be verified, is equivalent to

$$\int_{-\infty}^{\infty} |(e^{-\delta u} - e^{-\eta u}) e^{-u} G(\exp - e^{-u})|^2 \, du \to 0 \text{ as } (\delta, \eta) \to (0, 0).$$

Since this condition is identical with

$$\int_{0}^{\infty} x \, |(x^\delta - x^\eta) G(e^{-x})|^2 \, dx \to 0 \text{ as } (\delta, \eta) \to (0, 0),$$

it is satisfied if

$$\int_{0}^{\infty} x \, | G(e^{-x})|^2 \, dx < \infty, \text{ i.e., } \int_{0}^{1} (-\log r) \, | G(r)|^2 \, dr/r < \infty$$

$(r = e^{-x})$. But, since $-\log r \sim 1 - r$ as $r \to 1$, the convergence of the last integral is equivalent to (76) (there is no trouble at $r = 0$, since, according to (75),

$$(-\log r) \mid G(r) \mid {}^2/r = (-\log r) O(r^2)/r \text{ as } r \to 0,$$

and this is $O(-r \log r) \to 0$).

14. This proves (II), and therefore all theorems formulated above. Actually, the proof is such as to lead, without additional effort, to certain variants of these theorems. This is illustrated by the following extension of (III):

(IV) *If* $\lambda > 1$ *and* $\mu = \lambda/\lambda - 1)$, *then* (III) *remains true when its assumption* (81) *is generalized to*

$$\int_0^{1-0} (1-r)^{\mu/\lambda} \mid P(r) \mid {}^{\mu} \, dr < \infty \tag{88}$$

(which becomes (81) when $\lambda = 2 = \mu$).

In order to see this, it is sufficient to observe that the preceding application of Plancherel's theorem can be replaced by that of its Hölder form (Young, Hausdorff, Titchmarsh). Then all that remains to be ascertained is that, just as in the particular case (81), the (absolute) convergence of (76) for $\sigma > 1$ is aways implied by (88), where $P = G$. Thus it is seen from the formula line following (77), where $\sigma = 1 + \epsilon$ (and $G = P$), that it is sufficient to prove

$$\int^{1-0} \mid \log r \mid^{\epsilon} \mid P(r) \mid dr/r < \infty, \text{ i.e., } \int^{1-0} (1-r)^{\epsilon} \mid P(r) \mid dr < \infty,$$

for every $\epsilon > 0$. But this follows from (88) in the same way as in the above case, $\lambda = 2 = \mu$, of Schwarz's inequality.

In view of the classical case, (β), of $(\beta \text{ bis})$ in (vi), the following limiting form of (IV) is of particular interest:

(V) *Theorem* (III) *remains true if its assumption* (81) *is replaced by*

$$\int_0^{1-0} |P(r)| \, dr < \infty \tag{89}$$

(which is (88) with $\mu = 1 + 0$, $\lambda = \mu/(\mu - 1) = \infty$); in fact, (89) implies that

$$\psi(s) = \int_0^\infty x^{s-1} P(e^{-x})\,dx \tag{90}$$

is absolutely convergent at $s = 1$, and so the analytic function defined by (90) in the half-plane $\sigma > 1$ goes over into a continuous boundary function on the line $\sigma = 1$.

It is true that, in the limiting case (89) of (88), the (L^p)-theory of Fourier transforms does not apply. But such a theory is not needed now, since what is required by (β bis) in (vi) is trivial from the continuity of (90) on the closed half-plane $\sigma \geqq 1$. Hence, (V) follows from the proof of (vi) for the same reason as (x) did.

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ON SKEW-METRIC SPACES AND FUNCTION GROUPS.*

By Hwa-Chung Lee.

Introduction. In two previous papers [1] we have studied the even-dimensional spaces equipped with a nonsingular skewsymmetric fundamental tensor. We shall extend the consideration of such spaces to the cases where the fundamental tensor is not necessarily nonsingular, and the dimensionality may also be odd. We shall generalise the study of function groups to the *flat* spaces of this nature, and discover that the "group spaces" of function groups are natural examples of such flat spaces.

Let L_m be an m-dimensional space endowed with a skewsymmetric *covariant* fundamental tensor $a_{\alpha\beta}(x)$,[2] not necessarily nonsingular, whose components in each coordinate system are (differentiable) functions of the coordinates $x^1, \cdots, x^m$. The rank of the matrix $a_{\alpha\beta}$ being necessarily even, let it be $2n$, which we also call the *rank* of L_m.

The skewsymmetric quantity

$$K_{\alpha\beta\gamma} = \frac{\partial a_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial a_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial a_{\alpha\beta}}{\partial x^\gamma} \tag{1}$$

is a covariant tensor, which we call the *curvature tensor* of L_m. If this curvature tensor vanishes, L_m is said to be *flat*. For a flat L_m, we may use results in the Pfaff Problem to prove the existence of a special coordinate system in which the exterior form $a_{\alpha\beta}[dx^\alpha dx^\beta]$ reduces to [3]

$$2[dx^1 dx^2] + 2[dx^3 dx^4] + \cdots.$$

Hence, in this special coordinate system, the matrix whose element in the α-th row and β-th column is $a_{\alpha\beta}$ has the form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dotplus (\quad) \tag{2}$$

* Received December 1, 1946.

[1] H. C. Lee, "A kind of even-dimensional differential geometry and its application to exterior calculus," *American Journal of Mathematics*, vol. 65 (1943), pp. 433-438; "On even-dimensional skew-metric spaces and their groups of transformations," *ibid.*, vol. 67 (1945), pp. 321-328. These will be referred to as (I) and (II).

[2] The indices $\alpha, \beta, \gamma, \rho, \sigma, \tau$ run through the range $1, \cdots, m$.

[3] Cf. (I), p. 434. The reasoning applies whether $a_{\alpha\beta}$ is singular or nonsingular.

where the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is repeated n times, and the last direct summand ( ) denotes the zero matrix of order $m - 2n$. By rearrangement of rows and columns we have

THEOREM 1. *For a flat space L_m of rank $2n$, there exists a "canonical" coordinate system in which the fundamental tensor $a_{\alpha\beta}$ has constant components given by a matrix of the form*

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \dot{+} (\quad), \tag{3}$$

where I and 0 denote the unit and zero matrices of order n respectively, and ( ) is the zero matrix of order $m - 2n$.

If m is even and if L_m is of rank m, the covariant skewsymmetric tensor $a_{\alpha\beta}$ being nonsingular has an inverse $a^{\alpha\beta}$ which is a contravariant skewsymmetric tensor. But if $a_{\alpha\beta}$ is singular (certainly so if m is odd), $a^{\alpha\beta}$ does not exist. We are therefore led to introduce skew-metric spaces of another kind as follows.

1. The space L^m. Let there be given a skewsymmetric *contravariant* tensor $a^{\alpha\beta}(x)$, of rank $2n$ say. An m-dimensional space equipped with such a fundamental tensor is called an L^m of *rank* $2n$. The skewsymmetric quantity

$$K^{\alpha\beta\gamma} = a^{\alpha\rho}\frac{\partial a^{\beta\gamma}}{\partial x^\rho} + a^{\beta\rho}\frac{\partial a^{\gamma\alpha}}{\partial x^\rho} + a^{\gamma\rho}\frac{\partial a^{\alpha\beta}}{\partial x^\rho} \tag{4}$$

is a contravariant tensor, which we call the *curvature tensor* of L^m.

When (and only when) m is even and $2n = m$, the space L^m, of rank m, is said to be *nonsingular*. In this case $a^{\alpha\beta}$ has an inverse $a_{\alpha\beta}$, and the space L^m is at the same time a space L_m. The two tensors (1) and (4) are now related to each other by the equations [4]

$$K^{\alpha\beta\gamma} = a^{\alpha\rho}a^{\beta\sigma}a^{\gamma\tau}K_{\rho\sigma\tau}, \qquad K_{\alpha\beta\gamma} = a_{\alpha\rho}a_{\beta\sigma}a_{\gamma\tau}K^{\rho\sigma\tau}.$$

In any case, a space L^m is said to be *flat* if its curvature tensor $K^{\alpha\beta\gamma}$ vanishes:

$$a^{\alpha\rho}\frac{\partial a^{\beta\gamma}}{\partial x^\rho} + a^{\beta\rho}\frac{\partial a^{\gamma\alpha}}{\partial x^\rho} + a^{\gamma\rho}\frac{\partial a^{\alpha\beta}}{\partial x^\rho} = 0. \tag{5}$$

The question arises whether there exist special coordinate systems for a flat L^m, in which the fundamental tensor $a^{\alpha\beta}$ has constant components. The

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[4] See (II), p. 324.

answer is immediate in case m is even and L^m is nonsingular: for then L^m is also an L_m, the fundamental tensors $a^{\alpha\beta}$ and $a_{\alpha\beta}$ of L^m and L_m being inverse to each other, and by Theorem 1 there exists a canonical coordinate system of L_m in which $a_{\alpha\beta}$ is given by

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{6}$$

whence, in this coordinate system, $a^{\alpha\beta}$ is

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{7}$$

However, this proof is invalid in case L^m is singular. We shall show that for any flat L^m, the answer to the above question is in the affirmative.

2. Some properties of flat spaces. In passing we note certain formulas which will be of use later on. Consider any space L^m, not necessarily flat. For any two functions $f(x)$, $g(x)$ of the coordinates we form the (generalised) *Poisson expression*

$$(f, g) = a^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial g}{\partial x^\beta}, \tag{8}$$

which is skewsymmetric in f and g:

$$(f, g) = -(g, f).$$

For any three functions $f(x)$, $g(x)$, $h(x)$, it is easy to verify the relation

$$(f, (g, h)) + (g, (h, f)) + (h, (f, g)) = K^{\alpha\beta\gamma} \frac{\partial f}{\partial x^\alpha} \frac{\partial g}{\partial x^\beta} \frac{\partial h}{\partial x^\gamma}. \tag{9}$$

Hence, *when L^m is flat, the Jacobi identity*

$$(f, (g, h)) + (g, (h, f)) + (h, (f, g)) = 0 \tag{10}$$

is valid. If we define the operators

$$\Delta^\alpha = a^{\alpha\beta} \frac{\partial}{\partial x^\beta}, \tag{11}$$

called the *fundamental operators* of the space L^m, we find

$$\begin{aligned} \Delta^\alpha \Delta^\beta - \Delta^\beta \Delta^\alpha &= a^{\alpha\rho} \frac{\partial}{\partial x^\rho}\left(a^{\beta\sigma} \frac{\partial}{\partial x^\sigma}\right) - a^{\beta\sigma} \frac{\partial}{\partial x^\sigma}\left(a^{\alpha\rho} \frac{\partial}{\partial x^\rho}\right) \\ &= \left(a^{\alpha\rho} \frac{\partial a^{\beta\gamma}}{\partial x^\rho} + a^{\beta\rho} \frac{\partial a^{\gamma\alpha}}{\partial x^\rho}\right) \frac{\partial}{\partial x^\gamma} \\ &= \left(K^{\alpha\beta\gamma} - a^{\gamma\rho} \frac{\partial a^{\alpha\beta}}{\partial x^\rho}\right) \frac{\partial}{\partial x^\gamma}, \end{aligned}$$

and so we have the formula

$$\Delta^\alpha\Delta^\beta - \Delta^\beta\Delta^\alpha = \frac{\partial a^{\alpha\beta}}{\partial x^\gamma}\Delta^\gamma + K^{\alpha\beta\gamma}\frac{\partial}{\partial x^\gamma}. \tag{12}$$

In particular, *when L^m is flat, we have the relation*

$$\Delta^\alpha\Delta^\beta - \Delta^\beta\Delta^\alpha = \frac{\partial a^{\alpha\beta}}{\partial x^\gamma}\Delta^\gamma, \tag{13}$$

which states that the commutators of the operators Δ^α are linear combinations of these operators themselves. Hence *the m fundamental operators* $\Delta^1, \cdots, \Delta^m$ *of a flat space L^m constitute a "complete system."*

3. Canonical coordinate systems of a flat space L^m. We are now in position to show that for any flat space L^m we can determine a coordinate system in which the components $a^{\alpha\beta}$ of the fundamental tensor are all constants.

If all the $a^{\alpha\beta}$ vanish, there is no more to prove.

In the other case, take any $a^{\rho\sigma} \neq 0$ where ρ and σ are fixed, $\rho \neq \sigma$. If neither ρ nor σ is 1, we interchange the roles of the coordinates x^ρ and x^1. In other words, we can always choose the *new coordinate* x^1 (here by a mere renumbering of the coordinates) such that $a^{1\beta} \neq 0$ for at least a value of β $(\neq 1)$.

Then we set up the equation

$$a^{1\beta}\frac{\partial f}{\partial x^\beta} = -1,$$

whose solutions do not depend on x^1 alone (since $a^{11} = 0$). Choosing any solution of this equation as the *new coordinate* x^2, we have $a^{1\beta}\dfrac{\partial x^2}{\partial x^\beta} = -1$ so that

$$a^{12} = -1.$$

Consider now the two equations

$$a^{1\beta}\frac{\partial f}{\partial x^\beta} = 0, \qquad a^{2\beta}\frac{\partial f}{\partial x^\beta} = 0$$

which are independent since $a^{11} = 0$, $a^{12} = -1$, $a^{21} = 1$, $a^{22} = 0$; they can be solved for the two derivatives $\dfrac{\partial f}{\partial x^2}$ and $\dfrac{\partial f}{\partial x^2}$ as linear combinations of the remaining derivatives. The corresponding operators are Δ^1 and Δ^2, and their commutator is by (13)

$$\Delta^1\Delta^2 - \Delta^2\Delta^1 = \frac{\partial(-1)}{\partial x^\nu}\Delta^\gamma = 0.$$

Hence the two equations in question constitute a complete system, which, therefore, has $m-2$ independent solutions. Let f^p $(p=3,\cdots,m)$ be any set of such solutions. These functions being independent, the matrix

$$\left(\frac{\partial f^p}{\partial x^1},\frac{\partial f^p}{\partial x^2},\cdots,\frac{\partial f^p}{\partial x^m}\right)$$

of $m-2$ rows and m columns is of rank $m-2$. Since the first two columns of this matrix are linear combinations of the remaining $m-2$ columns, these $m-2$ columns must have a nonzero determinant, which is the functional determinant $\dfrac{\partial(f^3,\cdots,f^m)}{\partial(x^3,\cdots,x^m)}$. Then

$$\frac{\partial(x^1,x^2,f^3,\cdots,f^m)}{\partial(x^1,x^2,x^3,\cdots,x^m)}=\frac{\partial(f^3,\cdots,f^m)}{\partial(x^3,\cdots,x^m)}\neq 0,$$

and therefore the m functions $x^1,x^2,f^3,\cdots,f^m$ are independent. Hence it is justified to choose $f^3,\cdots,f^m$ as the *new coordinates* $x^3,\cdots,x^m$. If in the conditions

$$a^{1\beta}\frac{\partial f^p}{\partial x^\beta}=0,\qquad a^{2\beta}\frac{\partial f^p}{\partial x^\beta}=0\qquad (p=3,\cdots,m)$$

we set $f^p=x^p$, we have

$$a^{1p}=0,\qquad a^{2p}=0\qquad (p=3,\cdots,m).$$

We recall then the identities (5). Giving the free indices α,β,γ various values we have that the components a^{pq} $(p,q=3,\cdots,m)$ are functions of the coordinates $x^3,\cdots,x^m$ only, independent of x^1 and x^2, and that they satisfy the identities

$$a^{ps}\frac{\partial a^{qr}}{\partial x^s}+a^{qs}\frac{\partial a^{rp}}{\partial x^s}+a^{rs}\frac{\partial a^{pq}}{\partial x^s}=0\qquad (p,q,r,s=3,\cdots,m).$$

Consider a space of $m-2$ dimensions with the coordinates $x^3,\cdots,x^m$. A transformation of coordinates in this space is effected by $m-2$ independent functions

$$\bar{x}^3=\phi^3(x^3,\cdots,x^m),\cdots,\bar{x}^m=\phi^m(x^3,\cdots,x^m).$$

It is easily seen that the components a^{pq} $(p,q=3,\cdots,m)$ constitute a contravariant tensor in this space. In fact, if in the known transformation law

$$\bar{a}^{\rho\sigma}=a^{\alpha\beta}\frac{\partial\bar{x}^\rho}{\partial x^\alpha}\frac{\partial\bar{x}^\sigma}{\partial x^\beta}$$

of $a^{\alpha\beta}$ we restrict the free indices ρ and σ to the range $3,\cdots,m$, we have

$$\bar{a}^{rs} = a^{pq} \frac{\partial \bar{x}^r}{\partial x^p} \frac{\partial \bar{x}^s}{\partial x^q} \qquad (p, q, r, s = 3, \cdots, m).$$

This proves the tensor character of a^{pq} in the space considered. This space, equipped with the skewsymmetric contravariant tensor a^{pq}, is an L^{m-2}. The curvature tensor of this L^{m-2} vanishes because of the identities satisfied by a^{pq}, whence the space L^{m-2} is flat.

For this flat L^{m-2}, the whole of the above proof may be repeated, with the result that either all $a^{pq} = 0$ or we can determine a new coordinate system in L^{m-2} for which

$$a^{34} = -1, \qquad a^{3t} = a^{4t} = 0 \qquad (t = 5, \cdots, m).$$

Continuing this method of proof we finally arrive at a coordinate system in which the components of $a^{\alpha\beta}$ are given by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \dotplus (\quad). \tag{14}$$

If $a^{\alpha\beta}$ is of rank $2n$, the 2-rowed matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ occurs n times, and the last term $(\quad)$ is the zero matrix of order $m - 2n$. By a permutation of the coordinates we have the following theorem, analogous to Theorem 1:

THEOREM 2. *For a flat space L^m of rank $2n$, there exists a canonical coordinate system in which the fundamental tensor $a^{\alpha\beta}$ is given by*

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \dotplus (\quad), \tag{15}$$

where I and 0 denote the unit and zero matrices of order n respectively, and $(\quad)$ is the zero matrix of order $m - 2n$.

We note that in the nonsingular case, we have two different proofs of the same results (6) and (7).

4. Equivalence of flat spaces. Two spaces L^m (or two spaces L_m) are said to be *equivalent* if the fundamental tensor of one can be transformed into the fundamental tensor of the other by a change of coordinates. In consequence of Theorems 1 and 2, we have

THEOREM 3. *Two flat spaces L^m (or two flat spaces L_m) are equivalent if and only if they are of the same rank.*

5. Function groups. Consider a space L^m. Let there be r $(\leqq m)$

independent functions $y^1, \cdots, y^r$ of the m coordinates $x^1, \cdots, x^m$ such that the "Poisson parentheses"

$$b^{\lambda\mu} = (y^\lambda, y^\mu)\ ^5 \tag{16}$$

are functions of the functions $y^1, \cdots, y^r$ themselves. If $f(y)$ and $g(y)$ are any two functions of these functions y, we find by (8)

$$(f, g) = b^{\lambda\mu} \frac{\partial f}{\partial y^\lambda} \frac{\partial g}{\partial y^\mu} \tag{17}$$

which is again a function of the functions y. Hence the totality of functions of the functions y is closed with respect to the operation of Poisson parentheses in L^m, and is then called a *function group* of order r in L^m. The r independent functions $y^1, \cdots, y^r$ constitute a *basis* of the group, and any r *independent* functions

$$\begin{cases} \bar{y}^1 = \phi^1(y^1, \cdots, y^r) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \bar{y}^r = \phi^r(y^1, \cdots, y^r) \end{cases} \tag{18}$$

of the group also constitute a basis. The Poisson parentheses

$$\bar{b}^{\kappa\nu} = (\bar{y}^\kappa, \bar{y}^\nu)$$

of the new basis functions $\bar{y}^1, \cdots, \bar{y}^\nu$ are, by (17), given by

$$\bar{b}^{\kappa\nu} = b^{\lambda\mu} \frac{\partial \bar{y}^\kappa}{\partial y^\lambda} \frac{\partial \bar{y}^\nu}{\partial y^\mu}. \tag{19}$$

6. Group spaces. Any r functions of the group, arranged in a definite succession, may be regarded as the "coordinates" of a "point" in an r-dimensional space. We call this space the *group space* of the given group. A change of basis (18) of the group is a transformation of coordinates in the group space. It follows from the transformation law (19) that the quantities $b^{\lambda\mu}$ (skewsymmetric by definition) are the components of a *contravariant tensor* in the group space. Introducing this tensor as the fundamental tensor of the group space, the latter is an L^r.

The curvature tensor of this group space L^r, referred to the coordinate system $\{y^1, \cdots, y^r\}$, is similarly defined to be

$$R^{\lambda\mu\nu} = b^{\lambda\kappa} \frac{\partial b^{\mu\nu}}{\partial y^\kappa} + b^{\mu\kappa} \frac{\partial b^{\nu\lambda}}{\partial y^\kappa} + b^{\nu\kappa} \frac{\partial b^{\lambda\mu}}{\partial y^\kappa}. \tag{20}$$

We shall obtain another expression for this curvature tensor.

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[5] The indices λ, μ, ν, κ run through the range $1, \cdots, r$.

Let us examine formula (17). Its left-hand member is the Poisson expression of f and g in L^m according to definition (8), and its right-hand member is, according to a similar definition, the Poisson expression of f and g in L^r. Hence

THEOREM 4. *For any two functions f and g in the group space L^r (which are also functions in the space L^m), the expression (f, g) has a double meaning, whether relative to the space L^m or to the group space L^r.*

This double meaning of the formation (f, g) has important consequences. Let f, g, h be any three functions in L^r, and consider the expression

$$(f,(g,h)) + (g,(h,f)) + (h,(f,g)).$$

Regarding it as relative to L^m, we have formula (9); and if we refer it to L^r we have the similar relation

$$(f,(g,h)) + (g,(h,f)) + (h,(f,g)) = R^{\lambda\mu\nu} \frac{\partial f}{\partial y^\lambda} \frac{\partial g}{\partial y^\mu} \frac{\partial h}{\partial y^\nu}. \tag{21}$$

Hence, comparing this with (9), we have

$$R^{\lambda\mu\nu} \frac{\partial f}{\partial y^\lambda} \cdot \frac{\partial g}{\partial y^\mu} \frac{\partial h}{\partial y^\nu} = K^{\alpha\beta\gamma} \frac{\partial f}{\partial x^\alpha} \frac{\partial g}{\partial x^\beta} \frac{\partial h}{\partial x^\gamma}. \tag{22}$$

If we set f, g, h in (22) equal to y^λ, y^μ, y^ν respectively, we find that the curvature tensor of the group space L^r is given by

$$R^{\lambda\mu\nu} = K^{\alpha\beta\gamma} \frac{\partial y^\lambda}{\partial x^\alpha} \frac{\partial y^\mu}{\partial x^\beta} \frac{\partial y^\nu}{\partial x^\gamma}. \tag{23}$$

In particular, when L^m is flat: $K^{\alpha\beta\gamma} = 0$, we have also $R^{\lambda\mu\nu} = 0$. Hence

THEOREM 5. *The group space of a function group of order r in a flat space L^m is a flat space L^r.*

7. Function groups in flat spaces. Consider a function group of order r in a flat space L^m. The group space L^r of the group is flat by Theorem 5. Hence, by Theorem 2, there exists a canonical coordinate system for L^r in which the fundamental tensor $b^{\lambda\mu}$ of L^r (defined by (16)) reduces to a form like (15). Consequently,

THEOREM 6. *For a function group of order r in a flat space L^m, there exists a canonical basis $\{y^1, \cdots, y^r\}$ such that the Poisson expressions (y^λ, y^μ) are given by a matrix of the form*

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \dotplus (\quad) \tag{24}$$

where I and 0 denote the unit and zero matrices of order s and ( ) is the zero matrix of order $r-2s$.[6]

The rank of the r-rowed skewsymmetric matrix (y^λ, y^μ) is $2s$, which we call the *rank* of the group. This is also the rank of the group space L^r.

Take any p $(<r)$ of the functions $y^1, \cdots, y^r$ of a canonical basis. The Poisson parenthesis of these functions being constants $(=1,0)$, they can be regarded as functions of the p functions themselves. Hence these p functions, being independent, constitute a basis of a function group of order p, which is a subgroup of the given group. We have then

THEOREM 7. *A function group in a flat space L^m contains subgroups of every lower order.*

The set of all functions of the coordinates x of L^m is evidently a function group of order m. Hence

THEOREM 8. *In a flat space L^m there exist function groups of every order $r=1, \cdots, m$.*

8. Induced operators. Consider a general space L^m and the group space L^r of a function group, as in **5** and **6**. With (11) we define the induced operators

$$\delta^\lambda = \frac{\partial y^\lambda}{\partial x^a} \Delta^a. \tag{25}$$

It is required to find the commutators $\delta^\lambda\delta^\mu - \delta^\mu\delta^\lambda$ of these operators. We proceed as follows.

We first note that for a function g of the coordinates x we have in view of (25) and (11)

$$\delta^\lambda g = (y^\lambda, g). \tag{26}$$

Hence

$$\begin{aligned}\delta^\lambda\delta^\mu g - \delta^\mu\delta^\lambda g &= \delta^\lambda(y^\mu, g) - \delta^\mu(y^\lambda, g)\\ &= (y^\lambda, (y^\mu, g)) - (y^\mu, (y^\lambda, g))\\ &= -(g, (y^\lambda, y^\mu)) + K^{a\beta\gamma}\frac{\partial y^\lambda}{\partial x^a}\frac{\partial y^\mu}{\partial x^\beta}\frac{\partial g}{\partial x^\gamma}\end{aligned}$$

by (9), and since

[6] This is a generalisation of a classical theorem. See L. P. Eisenhart, *Continuous groups of transformations*, p. 285.

$$-(g,(y^\lambda,y^\mu))=(b^{\lambda\mu},g)=\frac{\partial b^{\lambda\mu}}{\partial y^\nu}(y^\nu,g)=\frac{\partial b^{\lambda\mu}}{\partial y^\nu}\delta^\nu g$$

we obtain the formula

$$\delta^\lambda\delta^\mu-\delta^\mu\delta^\lambda=\frac{\partial b^{\lambda\mu}}{\partial y^\nu}\delta^\nu+K^{\alpha\beta\gamma}\frac{\partial y^\lambda}{\partial x^\alpha}\frac{\partial y^\mu}{\partial x^\beta}\frac{\partial}{\partial x^\gamma}. \tag{27}$$

If in particular the operand g (as function of the x) is a function of the functions y, we may write

$$\frac{\partial}{\partial x^\gamma}=\frac{\partial y^\nu}{\partial x^\gamma}\frac{\partial}{\partial x^\nu}$$

in (27) and on account of (23) we have

$$\delta^\lambda\delta^\mu-\delta^\mu\delta^\lambda=\frac{\partial b^{\lambda\mu}}{\partial y^\nu}\delta^\nu+R^{\lambda\mu\nu}\frac{\partial}{\partial y^\nu}. \tag{28}$$

It should be remarked that in this case the right-hand side of (26) may be written $(y^\lambda,y^\mu)\dfrac{\partial g}{\partial y^\nu}$, so that we have

$$\delta^\lambda=b^{\lambda\mu}\frac{\partial}{\partial y^\nu}. \tag{29}$$

Hence, *operating on functions of the y's, the δ's are the fundamental operators of the group space L^ν* in the same way as the Δ's are fundamental operators of the space L^m. With this in view, and recalling (12) in connection with (11), we could have written down formula (28) without proof.

In particular, *when L^m is flat* (then L^ν is also flat), (27) or (28) reduces to

$$\delta^\lambda\delta^\mu-\delta^\mu\delta^\lambda=\frac{\partial b^{\lambda\mu}}{\partial y^\nu}, \tag{30}$$

and so *the operators $\delta^1,\cdots,\delta^r$ constitutes a complete system* (whether for the space L^m or for the group space L^r).

9. The reciprocal group and the centrum of a function group in flat space. Suppose that the space L^m is flat. Consider then the complete system of equations

$$\delta^\lambda g=0 \qquad (\lambda=1,\cdots,r), \tag{31}$$

which are not necessarily independent. For every solution of this system we have $(y^\lambda,g)=0$ by (26), whence $(f,g)=0$ for all functions $f(y)$ of the function group. We say that g is *involution* or *commutative* with f.

If g is regarded as a function of the variables $x^1,\cdots,x^m$, the number k of independent equations in the system (31) is equal to the rank of the

product matrix $\dfrac{\partial y^\lambda}{\partial x^\alpha} a^{\alpha\beta}$ (see (25)); thus $k \leqq \min(r, 2n)$ where $2n$ is the rank of $a^{\alpha\beta}$. In particular, if L^m is nonsingular (i.e., $m = 2n$), then $k = r$. The system (31) has $m - k$ independent solutions, $g^p(x)$ $(p = 1, \cdots, m - k)$. For any two of these solutions we have

$$\begin{aligned}\delta^\lambda(g^p, g^q) &= (y^\lambda, (g^p, g^q)) \\ &= -(g^p, (y^\lambda, q^q)) + (g^q, (y^\lambda, g^p)) \\ &= -(g^p, \delta^\lambda g^q) + (g^q, \delta^\lambda g^p) \\ &= 0,\end{aligned}$$

so that (g^p, g^q) is again a solution of (31) and is therefore a function of the functions $g^1, \cdots, g^{m-k}$. It follows that these functions constitute the basis of a function group of order $m - k$. Hence

THEOREM 9. *A function group in a flat space L^m determines another function group, the "reciprocal group," consisting of functions in involution with every function of the given group.*

If g is regarded as a function of the functions $y^1, \cdots, y^r$ the number of independent equations in the system (31) is (see (29)) equal to $2s$, the rank of $b^{\lambda\mu}$. This system has $r - 2s$ independent solutions, $g^p(y)$ $(p = 1, \cdots, r - 2s)$, which are commutative with every function of the given function group. The functions $g^1, \cdots, g^{r-2s}$ being themselves functions of the group, they constitute the basis of a subgroup. Hence

THEOREM 10. *The "centrum" of a function group of order r and rank $2s$ in a flat space L^m, consisting of all those functions of the group which are commutative with every function of the group, is a subgroup of order $r - 2s$. This subgroup is the intersection of the given group and its reciprocal group.*

ACADEMIA SINICA.

STUDY OF A SURFACE BY MEANS OF CERTAIN ASSOCIATE RULED SURFACES IN AFFINE SPACE.*

By George Wu.

In this paper we shall deduce some results concerning the theory of surfaces in affine space. Some of these results may be regarded as analogues of the projective theory of surfaces immersed in ordinary space. Others seem to be properties of surfaces immersed in affine space. In what follows we shall use the same notations as in Blaschke's *Vorlesungen über Differentialgeometrie* II.

First, we define the canonical quadric Q at an ordinary point P of a surface σ in affine space by means of the Bompiani-Klouboucek's asymptotic osculating quadrics. Using this quadric Q we give a simple geometrical interpretation of the Pick invariant J and the Gaussian curvature S of the fundamental quadric form $\phi = 2Fdudv$ of σ. In particular if $S = 0$, Q is a paraboloid, and conversely.

Next we study the Moutard quadrics $Q_n^{(u)}$ and $Q_n^{(v)}$ belonging to the tangent t_n and the asymptotic ruled surfaces $R^{(u)}$ and $R^{(v)}$ generated by the asymptotic u- and v-tangents along the v- and u-curves respectively. Using these two quadrics $Q_n^{(u)}$ and $Q_n^{(v)}$, we give a characteristic property of the tangents of Segre. We have shown that the loci of the diameters $d_n^{(u)}$ and $d_n^{(v)}$ of $Q_n^{(u)}$ and $Q_n^{(v)}$ are two quadric cones $\Gamma_2^{(u)}$ and $\Gamma_2^{(v)}$ passing respectively through the asymptotic u- and v-tangents and having the affine surface normal as a common generator. The cones $\Gamma_2^{(u)}$ and $\Gamma_2^{(v)}$ intersect the quadric of Lie in the asymptotic tangents and in a pair of non-composite twisted cubics $C_3^{(u)}$ and $C_3^{(v)}$. The tangent plane π of σ at P cuts the tangent surface of $C_3^{(u)}$ and $C_3^{(v)}$ in a pair of parabolas, mutually intersecting, besides the point P, and at three other points lying on the tangents of Segre. This property of the Segre tangents is similar to properties demonstrated by Cech [1] and Su.[2]

Analogous to a theorem of Transon we prove that the loci of the affine

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[1] E. Cech, "L'intorne di una superficie considerate dal punto di vista proiettivo," *Annali di Matematica*, vol. (3) 31 (1922), p. 205.

[2] B. Su, "The quadric of Moutard I," *Tohoku Mathematical Journal*, vol. 33 (1931), p. 35.

normals of the sections of $R^{(u)}$ and $R^{(v)}$ made by any plane π_{np} as it rotates about a non-asymptotic tangent t_n are two planes $\pi_n^{(u)}$ and $\pi_n^{(v)}$. As t_n varies, $\pi_n^{(u)}$ and $\pi_n^{(v)}$ envelop two cones which coincide with $\Gamma_n^{(u)}$, $\Gamma_n^{(v)}$. Moreover, the planes $\pi_n^{(u)}$, $\pi_n^{(v)}$ and the Transon plane T are concurrent in the conjugate tangent of t_n. The envelopes of the harmonic conjugate planes of π and T with respect to $\pi_n^{(u)}$ and $\pi_n^{(v)}$ are found. The planes $\pi_n^{(u)}$ and $\pi_n^{(v)}$ and those corresponding to the conjugate tangent t_{-n} are studied in detail.

1. Analytic basis. If we assume the asymptotic curves to be parametric then the coordinates $\mathfrak{x}$ of any non-parabolic point P of a non-ruled analytic surface σ satisfy the following system of completely integrable equations:

$$(1)\qquad \begin{cases} \mathfrak{x}_{uu} = (F_u/F)\mathfrak{x}_u + (A/F)\mathfrak{x}_v, \\ \mathfrak{x}_{uv} = F\mathfrak{y}, \\ \mathfrak{x}_{vv} = (D/F)\mathfrak{x}_u + (F_v/F)\mathfrak{x}_v; \end{cases}$$

$$(2)\qquad \begin{cases} \mathfrak{y}_u = -H\mathfrak{x}_u + (A_v/F)\mathfrak{x}_v, \\ \mathfrak{y}_v = +(D_u/F)\mathfrak{x}_u - H\mathfrak{x}_v. \end{cases}$$

From (1) and (2) we have

$$\begin{cases} \mathfrak{x}_{uuu} = (F_{uu}/F)\mathfrak{x}_u + (A_u/F)A\mathfrak{y}, \\ \mathfrak{x}_{uuv} = -FH\mathfrak{x}_u + (A_v/F)\mathfrak{x}_v + F_u\mathfrak{y}, \\ \mathfrak{x}_{uvv} = (D_u/F)\mathfrak{x}_u - 2FH_v + F_v\mathfrak{y}, \\ \mathfrak{x}_{vvv} = (D_v/F)\mathfrak{x}_u + (F_{vv}/F)\mathfrak{x}_v + D\mathfrak{y}; \end{cases}$$

$$(3)\qquad \begin{cases} \mathfrak{x}_{uuuu} = (*)\mathfrak{x}_u + (*)\mathfrak{x}_v + 2A_u\mathfrak{y}, \\ \mathfrak{x}_{uuuv} = (*)\mathfrak{x}_u + (*)\mathfrak{x}_v + (A_v + F_{uu})\mathfrak{y}, \end{cases}$$

where (*) denotes terms we shall not need. Let us denote the coordinates of any point ($\mathfrak{z}$) in space by

$$\mathfrak{z} = \mathfrak{x} + x\mathfrak{x}_u + y\mathfrak{x}_v + z\mathfrak{y}.$$

Then we have

$$\begin{cases} x = +(1/F)(\mathfrak{z} - \mathfrak{x}, \mathfrak{x}_v, \mathfrak{y}), \\ y = -(1/F)(\mathfrak{z} - \mathfrak{x}, \mathfrak{x}_u, \mathfrak{y}), \\ z = +(1/F)(\mathfrak{z} - \mathfrak{x}, \mathfrak{x}_u, \mathfrak{x}_v). \end{cases}$$

2. The canonical quadric. Lane[3] has defined the canonical quadric by using Bompiani-Klouboucek's asymptotic osculating quadrics. In an

[3] E. P. Lane, *Projective Differential Geometry of Curves and Surfaces*, University of Chicago Press, p. 119. Ex. 38.

analogous manner we can obtain the canonical quadric at a point of a surface in affine space. The equations of the asymptotic osculating quadrics [4] corresponding to the curve C_λ through P immersed in the one-parameter family of curves,

$$dv - \lambda du = 0,$$

in affine space are

$$Q^{(v)} \quad Hz^2 - 2z + 2Fxy + 2(D/F)\lambda^2 xz + 2(D/F)\lambda yz$$
$$- (A/F^2)\{A/F + \lambda(A_u/F) + \lambda^2(\partial/\partial v)\log(A^2/F) - \lambda'\}z^2 = 0$$

and

$$Q^{(v)} \quad Hz^2 - 2z + 2Fxy + 2(D/F)\lambda^2 xz + 2(D/F)\lambda yz$$
$$- (D/F^2)\{(D/F)\lambda^3 + \lambda^2(D_v/F) + \lambda(\partial/\partial u)\log(D^2/F) + \lambda'\}z^2 = 0$$

where λ' is the total differential of λ with respect to u.

The equation of the osculating plane of C_λ at the point P is

$$2F\lambda(\lambda x - y) + \{A/F - (F_u/F)\lambda + (F_v/F)\lambda^2 - (D/F)\lambda^3 + \lambda'\}z = 0.$$

This plane intersects the quadric $Q^{(u)}$ in a conic. The locus of this conic as C_λ varies but remains touching a fixed tangent t_λ at P is the quadric whose equation is

$$(4) \quad \bar{Q}^{(u)}: \quad \lambda^3(Sz^2 - 2z + 2Fxy) - 4(A/F)\lambda^2 xz + 4(A/F)\lambda yz$$
$$- (A/F^2)\{2A/F + \lambda(\partial/\partial u)\log(A/F) + 2\lambda^2(\partial/\partial v)\log A\}z^2 = 0.$$

On replacing the quadric $Q^{(u)}$ by $Q^{(v)}$, we find another quadric:

$$(5) \quad \bar{Q}^{(v)}: \quad Sz^2 - 2z + 2Fxy + 4(D/F)\lambda^2 xz - 4(D/F)\lambda yz$$
$$- (D/F^2)\{(2D/F)\lambda^3 + \lambda^2(\partial/\partial u)\log D/F + 2\lambda(\partial/\partial u)\log D\}z^2 = 0.$$

When $\lambda \to \infty$ in (4) and $\lambda \to \infty$ in (5), both $\bar{Q}^{(u)}$ and $\bar{Q}^{(v)}$ approach the same quadric Q:

$$2Fxy - 2z + Sz^2 = 0$$

where

$$S = J + H = -\frac{1}{F}\frac{\partial^2}{\partial u \partial v}\log F,$$

S being the Gaussian curvature of $\phi = 2Fdudv$. It is well known that the quadric Q is the canonical quadric of σ at P. Hence we have the

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[4] B. Su, "A note on the affine differential geometry of a surface," *Japanese Journal of Mathematics*, vol. 9 (1932), p. 235.

THEOREM. *If $S=0$, Q is a paraboloid and conversely.*

The center $(\mathfrak{z})$ of the quadric Q is

$$\mathfrak{z}=\mathfrak{x}+(1/S)\mathfrak{y}. \tag{6}$$

That is to say the reciprocal of the affine distance p of the center $(\mathfrak{z})$ of Q from the point P is the Gaussian curvature S. Let us denote the affine distance of the center of the quadric of Lie F_2 whose equation is

$$2Fxy-2z+Hz^2=0 \tag{7}$$

from P by p'; then $p'=1/H$. The Pick invariant J of σ is equal to $1/p-1/p'$. It is easy to see that a necessary and sufficient condition for $Q\equiv F_2$ is $J=0$. That is to say: $Q\equiv F_2$ if and only if σ is a ruled surface.

From (6) we have

$$\mathfrak{z}_u=(J/S)\mathfrak{x}_u+(A_v/F^2S)\mathfrak{x}_v+(1/S)_u\mathfrak{y},$$

$$\mathfrak{z}_v=(D_u/F^2S)\mathfrak{x}_u+(J/S)\mathfrak{x}_v+(1/S)_v\mathfrak{y}.$$

Hence a necessary and sufficient condition for $(\mathfrak{z})$ to be a fixed point in space is that σ be a ruled affine sphere $(J=A_v=D_u=0)$.

3. The Moutard quadrics of $R^{(u)}$ and $R^{(v)}$. Let us consider the asymptotic ruled surface $R^{(u)}$ $[R^{(v)}]$ generated by the asymptotic u-tangent [v-tangent] along the v-curve [u-curve]. It is evident that the point

$$\bar{\mathfrak{z}}=\mathfrak{x}+\mathfrak{x}_v$$

lies on the u-tangent of $\mathfrak{x}$. Hence we may take the v-curve and the curve generated by $\bar{\mathfrak{z}}$ as the director curves of $R^{(u)}$. Thus we have

$$\mathfrak{x}(u+du,0)=\mathfrak{x}+\mathfrak{x}_u du+\mathfrak{x}_{uu}(du^2/2!)+\mathfrak{x}_{uuu}(du^3/3!)+\mathfrak{x}_{uuuu}(du^4/4!)+$$

$$\bar{\mathfrak{z}}(u+du,0)=\mathfrak{x}+\mathfrak{x}_v+(\mathfrak{x}_u+\mathfrak{x}_{uv}du+(\mathfrak{x}_{uu}+\mathfrak{x}_{uuv})(du^2/2!)$$

$$+(\mathfrak{x}_{uuu}+\mathfrak{x}_{uuuv})(du^3/3!)+(4).$$

If we write

$$\mathfrak{x}(u+du,0)=\mathfrak{x}+\xi^{(0)}\mathfrak{x}_u+\eta^{(0)}\mathfrak{x}_v+\zeta^{(0)}\mathfrak{y},$$

$$\bar{\mathfrak{z}}(u+du,0)=\mathfrak{x}+\xi^{(1)}\mathfrak{x}_u+\eta^{(1)}\mathfrak{x}_v+\zeta^{(1)}\mathfrak{y}$$

then in view of (3) we find

$$\begin{cases} \xi^{(0)} = du + \dfrac{F_u}{F}\dfrac{du^2}{2!} + \dfrac{F_{uu}}{F}\dfrac{du^3}{3!} + (4), \\ \eta^{(0)} = \dfrac{A}{F}\dfrac{du^2}{2!} + \dfrac{A_u}{F}\dfrac{du^3}{3!} + (4), \\ \zeta^{(0)} = A(du^3/3!) + 2A_u(du^4/4!) + (5); \end{cases}$$

$$\begin{cases} \xi^{(1)} = du + [(F_u/F) - FH](du^2/2!) + (3), \\ \eta^{(1)} = 1 + [A/F + A_v/F](du^2/2!) + (3), \\ \zeta^{(1)} = Fdu + F_u(du^2/2!) + [A + A_v + F_{uu}](du^3/3!) + (4). \end{cases}$$

The coordinates of any point on $R^{(u)}$ may be expressed in the form

$$(8) \qquad x = \frac{\xi^{(0)} + p\xi^{(1)}}{1+p}, \qquad y = \frac{\eta^{(0)} + p\eta^{(1)}}{1+p}, \qquad z = \frac{\zeta^{(0)} + p\zeta^{(1)}}{1+p}$$

where p is to be regarded as a parameter and x, y, z as current coordinates.

The equation of one of the planes $\pi_{n\rho}$ passing through the non-asymptotic tangent t_n:

$$z = y - nx = 0$$

is of the form

$$(9) \qquad z = \rho(y - nx)$$

where ρ is a parameter. We demand that the parameter p be such that the point (8) lies on the plane (9). Substituting (8) into (9) we get

$$(10) \qquad \begin{aligned} p = ndu &+ \{2n^2 + nF_u/F - A/F + 2nF/\rho\}du^2/2! \\ &+ \{6n^3 + n^2[6(F_u/F) - 3FH] + n[F_{uu}/F - 6A/F - 3A_v/F] \\ &- A_u/F - 2A/\rho + 6nF_u/\rho + 12n^2F/\rho + 6nF^2/\rho^2\}du^3/3! + (4), \end{aligned}$$

and

$$(11) \qquad (1+p)^{-1} = 1 - ndu + \{-nF_u/F + A/F - 2nF/\rho\}du^2/2! + (3).$$

Substituting (10) and (11) into (8) we get the expansion of x up to the third order, namely:

$$x = du + \frac{F_u}{F}\frac{du^2}{2!} + \{-3nFH + (F_{uu}/F)\}du^3/3! + (4);$$

and that of z up to fourth order namely:

$$\begin{aligned} z = nFdu^2 &+ \{6nF_u - 2A + 6n(F/\rho)\}du^3/3! \\ &+ \{n^2[-12F^2H] + n[8F_{uu} - 8A_v + 6(F^2_u/F)] - 2A_u - 8AF/\rho \\ &+ 36n(FF_u/\rho) + 24n(F^3/\rho)\}du^4/4! + (5). \end{aligned}$$

It is not difficult to express z as power series in x, in the form

$$z = a_2x^2 + a_3x^3 + a_4x^4 + (5), \tag{12}$$

wherein we have placed

$$a_2 = nF,$$

$$a_3 = 1/6[-2A + 6n(F^2/\rho)],$$

$$a_4 = 1/24[12n^2FH - 8nA_v - 2A_u + 6A(F_u/F) - 8A(F/\rho) + 24n(F^2/\rho^2)]$$

Evidently (12) is the projection of the section of $R^{(u)}$ made by the plane $\pi_{n\rho}$ on the xz-plane. The osculating conic of (12) at P is

$$-x^2 + z(\alpha x + \beta z + \gamma) = 0 \tag{13}$$

where α, β, γ have the values

$$\alpha = A/3nF^2 - 1/n\rho,$$

$$\beta = -\frac{1}{2}\frac{H}{nF} + A_v/3n^2F^3 + A_u/12n^3F^3 - AF_u/4n^3F^4$$
$$+ A^2/9n^4F^4 - A/3n^3F^2\rho,$$

$$\gamma = nF.$$

By eliminating ρ from (9) and (13) we get the equation of the Moutard quadric [10] of $R^{(u)}$ belonging to the tangent t_n at P in the form

$$Q_n{}^{(u)}:\ 36n^3F^2(z - Fxy) + 24n^2AF^2x^2 - 12nAF^2y^2$$
$$+ \{4A^2 - 3Fn[A(\partial/\partial u)\log(F^3/A) - 4nA_v$$
$$- 6n^2F(\partial^2/\partial u\partial v)\log F - 6n^2AD/F]\}z^2 = 0.$$

Similarly we get the Moutard quadric [10] of $R^{(v)}$ belonging to the tangent t_n at P in the form

$$Q_n{}^{(v)}:\ 36F^2(z - Fxy) - 12n^2DF^2xz + 24nDF^2yz$$
$$+ \{4n^2D^2 - 3F[n^2D(\partial/\partial v)\log F^3/D - 4nDu$$
$$- 6F(\partial^2/\partial u\partial v)\log F - 6AD/F]\}z^2 = 0.$$

The residual conic K of the intersection of $Q_n{}^{(u)}$ and $Q_n{}^{(v)}$ lies in the plane

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[10] For the projective equivalence of these two quadrics see my paper, "Systems of quadrics associated with a point of a surface I, II," *Duke Mathematical Journal*, vol. 10 (1943), pp. 499-513, 515-530.

$$
\begin{aligned}
&12n^2F^2(2A + n^3D)x - 12nF^2(A + 2n^3D)y \\
&\quad + \{4(A^2 - n^6D^2) - 3Fn[A(\partial/\partial u)\log F^3/A \\
&\quad - n^3D(\partial/\partial v)\log F^3/D + 4n^3D_u - 4A_v]\}z = 0.
\end{aligned}
$$

A necessary and sufficient condition that K touch the tangent t_n at P is

$$A - n^3D = 0.$$

THEOREM. *The conic K is tangent to t_n if and only if t_n is a tangent of Segre.*

4. Associate cones $\Gamma_2^{(u)}$ and $\Gamma_2^{(v)}$. The diameter $d_n^{(u)}$ of $Q_n^{(u)}$ at P has the equations

$$\begin{cases} 2Az + 3nF^2y = 0, \\ Az - 3n^2F^2x = 0. \end{cases}$$

Similarly we have the diameter $d_n^{(v)}$ of $Q_n^{(v)}$ at P:

$$\begin{cases} 2nDz - 3F^2x = 0, \\ n^2Dz + 3F^2y = 0. \end{cases}$$

The loci of $d_n^{(u)}$ and $d_n^{(v)}$ as t_n varies in a pencil with the center P are the two cones

$$\Gamma_2^{(u)}:\ 3F^2y^2 + 4Axz = 0$$

and

$$\Gamma_2^{(v)}:\ 3F^2x^2 + 4Dyz = 0$$

which, we shall call the *associate cones $\Gamma_2^{(u)}$ and $\Gamma_2^{(v)}$* respectively.

THEOREM. *The loci of $d_n^{(u)}$ and $d_n^{(v)}$ as t_n varies in a pencil with the center P are two quadric cones $\Gamma_2^{(u)}$ and $\Gamma_2^{(v)}$ passing respectively through the asymptotic u- and v-tangents with the affine surface normal as a common generator.*

A parametric representation of the quadric of Lie (7) is

$$(14) \qquad x = \frac{2\lambda}{H\lambda\mu + 2F}, \qquad y = \frac{2\mu}{H\lambda\mu + 2F}, \qquad z = \frac{2\lambda\mu}{H\lambda\mu + 2F}.$$

Therefore, beside the asymptotic tangent $z = y = 0$, the intersection of (14) and $\Gamma_2^{(u)}$ is a twisted cubic $C_3^{(u)}$ whose equations are

$$(15) \qquad x = \frac{3F^2\lambda}{3F^3 - 2AH\lambda^3}, \quad y = \frac{4A\lambda^2}{3F^3 - 2AH\lambda^3}, \quad z = \frac{4A\lambda^3}{3F^3 - 2AH\lambda^3}.$$

Similarly on replacing $\Gamma_2^{(u)}$ by $\Gamma_2^{(v)}$ we obtain another twisted cubic $C_3^{(v)}$ whose equations are

$$x = -\frac{4D\mu^2}{3F^3 - 2DH\mu^3}, \quad y = \frac{3F^2\mu}{3F^3 - 2DH\mu^3}, \quad z = -\frac{4D\mu^3}{3F^3 - 2DH\mu^3}.$$

Differentiating (15) with respect to λ we obtain

$$\frac{dx}{d\lambda} = \frac{3F^2(3F^3 + 4AH\lambda^3)}{(3F^3 - 2AH\lambda^3)^2}, \quad \frac{dy}{d\lambda} = \frac{-8A\lambda(3F^2 + AH\lambda)}{(3F^3 - 2AH\lambda^3)^2},$$

$$\frac{dz}{d\lambda} = \frac{-36AF^3\lambda^2}{(3F^3 - 2AH\lambda^3)^2}.$$

Hence the tangent surface of $C_3^{(u)}$ can be represented by

$$(16) \qquad \begin{cases} x = \dfrac{3F^2\lambda}{3F^3 - 2AH\lambda^3} + 3F^2\rho[3F^3 + 4AH\lambda^3], \\[2ex] y = -\dfrac{4A\lambda^2}{3F^3 - 2AH\lambda^3} - 8A\lambda\rho[3F^3 + 4AH\lambda^3], \\[2ex] z = -\dfrac{4A\lambda^3}{3F^3 - 2AH\lambda^3} - 36AF^3\lambda^2\rho. \end{cases}$$

The section of (16) made by the tangent plane $z = 0$ is then

$$x = (2/3F)\lambda, \qquad y = -(4A/9F^3)\lambda^2, \qquad z = 0$$

or

$$(17) \qquad x^2 + (F/A)y = 0, \qquad z = 0$$

which has the asymptotic tangent $x = z = 0$ for its diameter and touches the asymptotic tangent $y = z = 0$. Replacing $C_3^{(u)}$ by $C_3^{(v)}$ we find another parabola

$$(18) \qquad y^2 + (F/D)x = 0, \qquad z = 0.$$

The two parabolas (17) and (18) intersect in the point P, and in three other points on the straight lines

$$z = Ax^3 - Dy^3 = 0.$$

Thus we have the

THEOREM. *The sections of the tangent surfaces of $C_3^{(u)}$ and $C_3^{(v)}$ made by the tangent plane at P are two parabolas, each of which touches one of the asymptotic tangents and has the other asymptotic tangent as its affine normal. They intersect in the point P, and in three other points lying on a tangent of Segre.*

The polar planes of the tangent t_n with respect to $\Gamma_2^{(u)}$ and $\Gamma_2^{(v)}$ determine a line

$$\begin{cases} 3nF^2y + 4Az = 0, \\ 2F^2x + 4nDz = 0. \end{cases}$$

By eliminating n from the above equations we obtain the equation of a cone of the second order

$$9F^4xy - 4ADz^2 = 0.$$

We call this cone *the associate cone* $\Gamma_2^{(uv)}$. Thus we have the

THEOREM. *The polar line of the tangent plane of σ at P with respect to the associate cone $\Gamma_2^{(uv)}$ is the affine surface normal.*

5. An analogue of a theorem of Transon. Transon [5] has proved that all of the affine normals of the plane sections of σ made by $\pi_{n\rho}$ lie in a plane T. The equation of T is

$$3F^2(y + nx) - (A + Dn^3)z = 0.$$

This plane is the so called Transon Plane. Analogously we shall prove that the affine normals of the plane sections of $R^{(u)}$ $[R^{(v)}]$ made by $\pi_{n\rho}$ lie also in a plane $\pi_n^{(u)}[\pi_n^{(v)}]$. We call it *the associate plane* $\pi_n^{(u)}[\pi_n^{(v)}]$.

In order to prove the statement we adopt Salkowski's interpretation [6] of the affine normal of a plane curve. He has stated that the affine normal at a point of a plane curve is the diameter of the parabola osculating the curve at this point. It is easily shown that the osculating parabola of the curve (12) at the point P is of the form

$$z = (rx + sz)^2, \qquad y = 0 \tag{19}$$

wherein we have placed

$$r = \sqrt{nF}, \qquad s = 1/12r^3(-2A + 6nF^2/\rho).$$

Hence the affine normal of the curve (12) at P is

$$rx + (1/12r^3)(-2A + 6nF^2/\rho) = 0, \qquad y = 0. \tag{20}$$

By eliminating ρ from (9) and (20) we obtain the equation of the associate plane $\pi_n^{(u)}$ in the form

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[5] A. Transon, "Recherches sur la théorie des lignes et des surfaces," *Journal des Mathématiques* (1), vol. 6 (1841), pp. 191-208.

[6] E. Salkowski, *Affine Differentialgeometrie* (1935), S. 49-50.

$$3F^2n(y+nx)-Az=0. \tag{21}$$

Similarly the equation of the associate plane $\pi_n^{(v)}$ takes the form

$$3F^2(y+nx)-n^2Dz=0. \tag{22}$$

It can be shown that the associate plane $\pi_n^{(u)}[\pi_n^{(v)}]$ envelops a quadric cone which coincides with $\Gamma_2^{(u)}[\Gamma_2^{(v)}]$.

THEOREM. *The affine normals of the sections of $R^{(u)}[R^{(v)}]$ made by planes $\pi_{n\rho}$ at P lie in a plane $\pi_n^{(u)}[\pi_n^{(v)}]$. The envelope of this plane $\pi_n^{(u)}[\pi_n^{(v)}]$ is the associate cone $\Gamma_2^{(u)}[\Gamma_2^{(v)}]$.*

This may be taken as another construction of the $\Gamma_2^{(u)}[\Gamma_2^{(v)}]$. From this theorem we can easily write $\Gamma_2^{(u)}$ and $\Gamma_2^{(v)}$ in plane coordinates in the form

$$3F^2u_1u_3+Au_2^2=0$$

and

$$3F^2u_2u_3+D_1^2=0$$

respectively.

By eliminating ρ between (9) and (19) we obtain the locus of the osculating parabola (19) as a parabolic cylinder whose equation is of the form

$$z=(1/36n^3F^3)[3nF^2(y+nx)-Az]^2. \tag{23}$$

Similarly by replacing $R^{(u)}$ by $R^{(v)}$ we get another parabolic cylinder

$$z=(1/36n^3F^3)[3F^2(y+nx)-n^2Dz]^2. \tag{24}$$

It is easily seen that the diametral planes of (23) and (24) passing through P are (21) and (22) respectively. Hence we have the

THEOREM. *The locus of the osculating parabolas of the sections of $R^{(u)}[R^{(v)}]$ made by planes $\pi_{n\rho}$ is a parabolic cylinder whose diametral plane coincides with $\pi_n^{(u)}[\pi_n^{(v)}]$ and consequently envelops the cone $\Gamma_2^{(u)}[\Gamma_2^{(v)}]$.*

The above theorem is similar to a theorem of Kubota[7] in which he has proved that the envelope of the Transon plane T, as t_n varies in a pencil, is a cone Γ_4 found by Su.

It is easily seen that the Transon plane T, the associate planes $\pi_n^{(u)}$ and $\pi_n^{(v)}$ belong to a pencil whose axis is the line t_{-n}.

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[7] T. Kubota, "Einige Bemerkungen zur Affinflächentheorie, *Science Reports Tohoku Imperial University* (1), vol. 19 (1930), pp. 163-168.

THEOREM. *The associate planes $\pi_n^{(u)}$ and $\pi_n^{(v)}$, and Transon plane T, are concurrent on the conjugate tangent of t_n.*

6. Further associate cones. The harmonic conjugate plane $\bar{T}$ of the Transon plane T with respect to $\pi_n^{(u)}$ and $\pi_n^{(v)}$ has the equation

$$3F^2n(A+n^3D)(y+nx)-(A^2+n^6D^2)z=0,$$

or in plane coordinates

$$\begin{cases}\rho u_1=3F^2n^2(A+n^3D),\\ \rho u_2=3F^2n(A+n^3D),\\ \rho u_3=-(A^2+n^6D^2).\end{cases} \tag{25}$$

On eliminating ρ, n from (25) we obtain the equation of the envelope of $\bar{T}$ in the form

$$A^2u_2+D^2u_1^6+3F^2u_1u_2u_3(Au_2^3+Du_1^3)=0.$$

We state our result in the

THEOREM. *The harmonic conjugate plane $\bar{T}$ of the Transon plane T with respect to the associate planes $\pi_n^{(u)}$ and $\pi_n^{(v)}$ envelops an algebraic cone of class six.*

We now consider the plane $\bar{\pi}$ conjugate to the plane π with respect to the associate planes $\pi_n^{(u)}$ and $\pi_n^{(v)}$. Hence the equation of $\bar{\pi}$ can be written in the form

$$3F^2(A+n^3D)(y+nx)-2ADn^2z=0$$

or

$$\begin{cases}\rho u_1=3F^2n(A+n^3D),\\ \rho u_2=3F^2(A+n_3D),\\ \rho u_3=-2ADn^2.\end{cases}$$

The envelope of this plane as t_n varies is

$$3u_3(Au_2^3+Du_1^3)+2FJu_1^2u_2^2=0.$$

THEOREM. *The envelope of the plane $\bar{\pi}$, conjugate to the tangent plane π with respect to the associate planes $\pi_n^{(u)}$ and $\pi_n^{(v)}$ belonging to a variable tangent t_n, is an algebraic cone of class four.*

Now the equation of the harmonic conjugate plane $\bar{\omega}_1$ of π with respect to the associate plane $\pi_n^{(u)}$ and the Transon plane T can be written in the form

$$3nF^2(2A+n^3D)(y+nx)-2A(A+n^3D)z=0$$

or

$$\begin{cases}\rho u_1 = 3n^2F^2(2A+n^3D),\\ \rho u_2 = 3nF^2(2A+n^3D),\\ \rho u_3 = -2A(A+n^3D).\end{cases}$$

Hence the envelope of this plane has the equation

$$3F^2u_1u_3(2Au_2^3+Du_1^3)+2A_2^2(Au_2^3+Du_1^3)=0.$$

Similarly, the plane $\tilde{\omega}_2$ determined by

$$(\pi_n{}^{(v)}T\tilde{\omega}_2\pi)=-1$$

envelops a cone whose equation is

$$3F^2u_2u_3(Au_2^3+2Du_1^3)+2Du_1^2(Au_2^3+Du_1^3)=0.$$

Hence we may state the

THEOREM. *The harmonic conjugate plane $\tilde{\omega}_1[\tilde{\omega}_2]$ of the tangent plane π with respect to the associated plane $\pi_n{}^{(u)}[\pi_n{}^{(v)}]$ and the Transon plane T envelops an algebraic cone of class five.*

7. Further consideration of the associate planes $\pi_n{}^{(u)}$ and $\pi_n{}^{(v)}$. From the last theorem of **5** we observe that the conjugate tangent t_{-n} plays an especially important rôle in the affine theory of surfaces. The associate planes $\pi_{-n}{}^{(u)}$ and $\pi_{-n}{}^{(v)}$ belonging to the conjugate tangent t_{-n} are worthy of consideration. The equations of $\pi_{-n}{}^{(u)}$ and $\pi_{-n}{}^{(v)}$ are

$$3nF^2(y-nx)+Az=0$$

and

$$3F^2(y-nx)-n^2Dz=0$$

respectively. Hence the line $l(\pi_n{}^{(u)}, \pi_{-n}{}^{(u)})$, the intersection of $\pi_n{}^{(u)}$ and $\pi_{-n}{}^{(u)}$, has the equations

$$\begin{cases}3n^2F^2x-Az=0,\\ y=0\end{cases}$$

and the line $l(\pi_n{}^{(v)}, \pi_{-n}{}^{(v)})$ has the equations

$$\begin{cases}3n^2F^2y-Dz=0,\\ x=0.\end{cases}$$

Evidently as t_n varies $l(\pi_n{}^{(u)}, \pi_{-n}{}^{(u)})$ and $l(\pi_n{}^{(v)}, \pi_{-n}{}^{(v)})$ describe two pencils

with the center P. The plane in which $l(\pi_n^{(u)}, \pi_{-n}^{(u)})$ and $l(\pi_n^{(v)}, \pi_{-n}^{(v)})$ lie has the equation

$$3n^2F^2(Dx + Ay) - ADz = 0. \tag{26}$$

This plane always passes through the tangent

$$z = Ay + Dx = 0.$$

As t_n varies the lines

$$l(\pi_n^{(u)}, \pi_{-n}^{(v)}) : \begin{cases} 6nF^2y - (A + n^3D)z = 0, \\ 6n^2F^2x - (A - n^3D)z = 0 \end{cases}$$

and

$$l(\pi_{-n}^{(u)}, \pi_n^{(v)}) : \begin{cases} 6nF^2x - (A + n^3D)z = 0, \\ 6n^2F^2y - (A - n^3D)z = 0 \end{cases}$$

describe the cones

$$162F^2x^2y^2 + 27(Ax^3 - Dy^3) - 9FJxyz^2 - J^2z^4 = 0$$

and

$$162F^2x^2y^2 + 27(Ay^3 - Dx^3) - 9FJxyz^2 - J^2z^4 = 0$$

respectively. These two cones are of order four and each has double contact with the tangent plane along the asymptotic tangents. The plane containing the lines $l(\pi_n^{(u)}, \pi_{-n}^{(v)})$ and $l(\pi_{-n}^{(u)}, \pi_n^{(v)})$ is

$$6n^2F^2(x + y) - \{(A - n^3D) + n(A + n^3D)\}z = 0, \tag{27}$$

which describes a pencil with the tangent

$$z = x + y = 0$$

as its axis. It can be easily verified that the line determined by the planes (27) and (26) generates an algebraic cone Γ_7 of order seven.

8. Further considerations of diameters $d_n^{(u)}$ and $d_n^{(v)}$. The plane $\pi(d_n^{(u)}, d_n^{(v)})$, containing $d_n^{(u)}$ and $d_n^{(v)}$ is given by the equation

$$n(2A + n^3D)x + (A + 2n^3D)y - n^2FJz = 0$$

which envelops an algebraic cone of order six and class four. The equation of this cone in plane coordinates is

$$4Ju_3(Au_2^3 + Du_1^3) + 18F^2Ju_1u_2u_3^2 + FJ^2u_1^2u_2^2 - 27F^3u_3^4 = 0. \tag{28}$$

The equation of the cone (28) in point coordinates is

$$27(A^2x^6+D^2y^6)+144F^3Jx^3y^3+54FJxyz(Ax^3+Dy^3)$$
$$-J^2z^3(Ax^3+Dy^3)-FJ^3xyz^4+15F^2J^2x^2y^2z^2=0.$$

Obviously the plane $\pi(d_{-n}{}^{(u)}, d_{-n}{}^{(v)})$, whose equation is

$$n(2A-n^3D)x-(A-2n^3D)y+n^2FJz=0,$$

also envelopes the cone (28). The line of intersection of $\pi(d_n{}^{(u)}, d_n{}^{(v)})$ and $\pi(d_{-n}{}^{(u)}, d_{-n}{}^{(v)})$ is $\begin{cases} n^4Dx+Ay-n^2FJz=0, \\ Dn^2y+Ax=0 \end{cases}$

which generates an algebraic cone Γ_3 of order three, whose equation is

$$Ax^3+Dy^3+FJxyz=0.$$

This cone intersects the tangent plane in the tangents of Darboux.

In a similar way we may show that the planes $\pi(d_n{}^{(u)}, d_{-n}{}^{(v)})$ and $\pi(d_{-n}{}^{(u)}, d_n{}^{(v)})$ envelop the same cone whose equation is

$$4Ju_3(Au_2{}^3+Du_1{}^3)-6F^2Ju_1u_2u_3{}^2-(5/3)FJu_1{}^2u_2{}^2-27F^3u_3{}^4=0,$$

and their line of intersection describes the cone

$$Ax^3+Dy^3+(5/3)\ FJxyz=0.$$

Finally the planes $\pi(d_n{}^{(u)}, d_{-n}{}^{(u)})$ and $\pi(d_n{}^{(v)}, d_{-n}{}^{(v)})$ pass through the line

$$\begin{cases} Az+3n^2F^2x=0, \\ n^2Dz+3F^2y=0, \end{cases}$$

which describes the cone

$$9F^4xy=ADz^2. \tag{29}$$

This cone has been studied first by Su [8] and subsequently by Kubota.[9] *The polar line of the tangent plane with respect to the cone* (29) *is the affine surface normal.*

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China.

[8] B. Su and A. Ichida, "On certain cones connected with a surface in affine space," *Japanese Journal of Mathematics*, vol. 10 (1930), p. 214.

[9] T. Kubota, "Einige Bemerkungen zur Affinflächentheorie," *Japanese Journal of Mathematics*, vol. 10 (1930), p. 216.

VORTICES AND NODES.*

By Aurel Wintner.

Let a, b, c, d be real constants with a determinant

(1) $$ad - bc \neq 0,$$

and let $f(x,y)$, $g(x,y)$ be real-valued functions, defined in a circle, say $r \leqq \alpha$, about the origin of an (x,y)-plane so as to be continuous and to satisfy

(2) $$f(x,y) = o(r), \qquad g(x,y) = o(r) \text{ as } r \to 0,$$

where

(3) $$x = r\cos\theta, \qquad y = r\sin\theta \qquad (r > 0).$$

The following considerations, which will always assume that the above conditions are fulfilled, will deal with the problem initiated (and, in the analytic case, solved) by Poincaré (cf., e. g., [1]), namely, with the problem of asymptotic connections between the solutions of the system

(4) $$x' = ax + by + f(x,y), \qquad y' = cx + dy + g(x,y)$$

and of the trivial system

(5) $$x' = ax + cy, \qquad y' = cx + dy,$$

where the primes denote differentiations with respect to a real variable, t.

1. Let the point $(x,y) = (0,0)$ be called an *attractor* of (4) if the above α can be replaced by a β $(\leqq \alpha)$ having the following property: If a solution path,

(6) $$x = x(t), \qquad y = y(t),$$

of (4) has at least one point in the circle $r < \beta$, then (6) tends to the point $(0,0)$ (when *either* $t \to \infty$ *or* $t \to -\infty$). The existence of (6) for a whole t-half-line is part of this requirement. In Poincaré's terminology, vortices (foci) and nodes (of any kind) are attractors in this sense, but whirls (centra) and saddle points are not. Since f and g, instead of being analytic, are arbitrary continuous functions satisfying (2), an attractor can

* Received March 25, 1947.

be neither a pure vortex nor a pure node. In addition, (4) can have more than one solution (6) passing through the same point (x_0, y_0), since the assumptions made before (3) do not imply anything like a Lipschitz condition. Nevertheless, the following fact is true under the assumptions preceding (3):

(i) *If* $(0,0)$ *is an attractor of* (5), *then it is an attractor of* (4) *also.*

The converse of (i) is false, since, even in Poincaré's analytic case, $(0,0)$ can be a vortex of (4) when it is a whirl of (5).

First, if a real matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{7}$$

has a positive determinant, and if $B(x,y)$ denotes the bilinear form belonging to (7), it is easy to verify (for instance, by considering the various affine normal forms of a real binary matrix), that, after a suitable affine transformation of the (x,y)-space, either the characteristic numbers of (7) are purely imaginary or else

$$\min_{x^2+y^2=1} B(x,y) > 0 \tag{8}$$

(whether the characteristic numbers be complex or real and, in the second case, whether (7) does or does not have a multiple elementary divisor). It follows that $(0,0)$ is an attractor of (5) [if and] only if a matrix equivalent to the matrix of (5) satisfies (8). In fact, if the determinant of (7) is negative, then the characteristic numbers of (8) are real and of opposite sign, and so the affine normal form of (5) is $x' = px$, $y' = -qy$, where $p > 0$ and $q > 0$. Similarly, if (7) has purely imaginary characteristic numbers (and so, in view of (1), a positive determinant), the affine normal form of (5) is $x' = sy$, $y' = -sx$, where $s \neq 0$. Hence, $(0,0)$ is a saddle point of (5) in the first case and a whirl of (5) in the second case, and so no attractor of (i) in either case. [It is clear, but for the present immaterial, that $(0,0)$ is either a node or a vortex, and therefore an attractor, of (5) in the remaining cases.] Accordingly, (i) will be proved if it is shown that $(0,0)$ must be an attractor of (4) if (7) satisfies (8).

To this end, let $r = r(t)$ in (6) refer to a solution (6) of (4). Since $xx' + yy' = rr'$, it is seen from (4) and from the definition of $B(x,y)$ that

$$rr' = B(x,y) + xf(x,y) + yg(x,y).$$

Hence, (2) and (8) imply that

$$rr' \geqq \lambda r^2 + o(r^2),$$

where λ is a positive constant and $r = r(t)$. It will be convenient to replace t by $-t$ (this is admissible, since (4) does not contain t explicitly). Then it is clear from the last formula line that

$$r' \leqq -\tfrac{1}{2}\lambda r \text{ whenever } r < \beta,$$

where λ and β are positive constants. But a well-known argument shows (cf. [3], Appendix) that the existence of two such positive constants λ, β implies the following fact: If a solution curve (6) of (5) is within the circle $r < \beta$ at *some* $t = t_0$, then this solution of (5) exists on the whole half-line $t_0 < t < \infty$ and is such that $r(t) \to 0$ holds as $t \to \infty$. Since this means that $(0, 0)$ is an attractor of (4), the proof of (i) is complete.

2. The term *node* was used above in its usual sense: The point $(0, 0)$ is a node of (4) if it is an attractor having the property that every solution path tending to $(0, 0)$ has a tangent at $(0, 0)$. Let $(0, 0)$ be called a *proper node* of (4) if it is a node and has, in addition, the property that every half-line issuing from $(0, 0)$ is the tangent at $(0, 0)$ of some solution path tending to $(0, 0)$.

In the trivial case (5), the point $(0, 0)$ is a node if (and only if) the characteristic numbers of (7) are real and of the same sign, but a proper node (if and) only if the characteristic numbers of (7) are equal and belong to distinct elementary divisors. [In fact, the prototype of (5) is $x' = x$, $y' = y$ in the latter case. This has the general solution $x = x_0 e^t$, $y = y_0 e^t$, which (besides the trivial solution determined by $x_0 = 0$, $y_0 = 0$) represents all half-lines issuing from $(0, 0)$. But there are two further cases of real characteristic numbers with common sign, namely, the cases in which the prototype of (5) becomes either $x' = x$, $y' = \lambda y$, where $0 < \lambda \neq 1$, or $x' = x$, $y' = x + y$. In these cases, the general solutions are $x = x_0 e^t$, $y = y_0 e^{\lambda t}$ and $x = x_0 e^t$, $y = (x_0 t + y_0) e^t$ respectively. Hence, $(0, 0)$ is a node in both cases, but is not a proper node in either case, since only a finite number of half-lines issuing from $(0, 0)$ become tangents in both cases.]

Since only the continuity of f and g is assumed in (2) and (5), nothing hinders that, when $(0, 0)$ is a proper node of (5), the correspondence between solution paths and half-lines through $(0, 0)$ be not one-to-one.

The following facts lie quite on the surface and are collected here only in order to contrast them with (iii) and (iv) below.

(ii a) *If* $(0, 0)$ *is a proper node of* (5), *it can be a vortex of* (4).

(ii b) *If* $(0, 0)$ *is a vortex of* (5), *it must be a vortex of* (4) *also.*

(ii c) *If* $(0, 0)$ *is a vortex of* (5), *and if the unit of length on the t-axis is so chosen that the real part of the characteristic numbers of* (7) *becomes of absolute value* 1, *then* $\log r(t) \sim -t$ *holds for every solution* (6) *of* (5) *satisfying* $r(t) = o(1)$.

Here, and in the sequel, the point $(0, 0)$ is called a *vortex* of (4) if it is an attractor of (4) and has the property that $|\theta(t)| \to \infty$ holds for every solution of (4) satisfying $0 < r(t) \to 0$; cf. (3) and (6).

Let $a = d = 1$ and $b = c = 0$. Then $(0, 0)$ is a proper node of (5), and (4) becomes

$$x' = x + f(x, y), \qquad y' = y + g(x, y). \tag{9}$$

The assertion of (ii a) is proved by an example of Perron [2], pp. 128-129, as follows: Choose, in terms of (3),

$$f(x, y) = -h(r) r \sin \theta, \qquad g(x, y) = h(r) r \cos \theta,$$

where $h = h(r)$ is any continuous function vanishing at $r = 0$. Then $f(x, y)$ and $g(x, y)$ are continuous functions satisfying (2). Clearly, $h(r)$ is identical with $(g(x, y) \cos \theta - f(x, y) \sin \theta)/r$ and therefore, by (3), with $(xg - yf)/r^2$ and so, by (9), with $(y'x - x'y)/r^2$, which is θ', by (3). But the definition of f and g (in terms of an h) also shows that $xf + yg$ vanishes identically, and so $xx' + yy' = xx + yy$, by (9). In view of (3), this means that $r' = r$, that is, $r = r_0 e^t$. Since $h(r)$ was seen to be identical with θ', it follows that $\theta(t) = \int h(r_0 e^t) dt$. Hence, it is sufficient to choose $h(r) = (\log r)^{-1}$ (if $r > 0$, and $h(0) = 0$), in order to see that $r(t) \to 0$ and $|\theta(t)| \to \infty$ (as $t \to -\infty$) are satisfied by every choice of both integration constants θ_0, r_0 (> 0). This proves (ii a).

The assertion of (ii b) is well-known and will be verified here only because a simple proof can be based on (i). In order to see this, let $(0, 0)$ be a vortex of (5). It can be assumed that (7) is in the normal form mentioned in (ii c). Then (5) appears in the form

$$x' = -x + \lambda y + f(x, y), \qquad y' = -y - \lambda x + g(x, y) \tag{10}$$

(after a suitable rotation of the (x, y)-plane, the characteristic numbers of (7) being $-1 \pm i\lambda$, where $\lambda \neq 0$, and so, without loss of generality, $\lambda > 0$). Since θ' is identical with $(y'x - x'y)/r^2$, by (3), and therefore, by (10) and (2), with $1/r^2$ times $0 - \lambda x^2 - \lambda y^2 + o(r^2)$,

$$r^2 \theta' = -\lambda r^2 + o(r^2). \tag{11}$$

But $(0, 0)$ is a vortex of (5), hence an attractor of (5), and so, by (i), an attractor of (4). Hence, $r(t) \to 0$ holds when either $t \to \infty$ or $t \to -\infty$. As will be seen in the proof of (ii c) below, the normalization (10) implies that $r(t) \to 0$ holds when $t \to \infty$. Since $\lambda > 0$, it now follows from (11) that $\theta' = -\lambda + o(1)$, hence $\theta = -\lambda t + o(t)$, and so $|\theta| \to \infty$, as $t \to \infty$. This proves (ii b).

Finally, from (10) and (3),

$$rr' = -r^2 + xf + yg. \tag{12}$$

Hence, $r' = -r + o(r)$, by (2). Since this means that $(\log r)' = -1 + o(1)$ as $t \to \infty$, the assertion of (ii c) follows.

3. If (4) is of the form (10), then (5) becomes

$$x' = -x + \lambda y, \qquad y' = -y - \lambda x \tag{13}$$

and has therefore the general solution

$$x = (v \cos \lambda t + u \sin \lambda t)e^{-t}, \qquad y = (v \sin \lambda t - u \cos \lambda t)e^{-t}, \tag{14}$$

where u and v are arbitrary integration constants. Since (14) implies that $r(t) = (x^2 + y^2)^{\frac{1}{2}}$ is identical with $r_0 e^{-t}$, where $r_0 = (u^2 + v^2)^{\frac{1}{2}}$ is arbitrary, one might expect that the (logarithmic) assertion of (ii c) can be refined to $r(t) \sim r_0 e^{-t}$, where r_0 (> 0) is an integration constant. But the example proving (iii bis) below will show that this refinement of (ii c) is false. However, it becomes true under the Tauberian restriction which replaces the $o(r)$ in (2) by $O(r^{1+\epsilon})$, where $\epsilon > 0$. This Tauberian fact (and somewhat more) is contained in the following theorem:

(iii) *If* $(0, 0)$ *is a vortex of* (5) *and if* (2) *in* (4) *is refined to*

$$f(x, y) = O(r^{1+\epsilon}), \qquad g(x, y) = O(r^{1+\epsilon}), \qquad (\epsilon > 0), \tag{15}$$

then every solution path (6) *of* (4) *tending to the vortex* $(0, 0)$ *of* (4) *is asymptotic to a solution path* (6) *of* (5), *and every solution path* (6) *of* (5) *is asymptotic to a solution path* (6) *of* (4).

It is not claimed that this asymptotic correspondence between the solutions of (4) and those of the trivial system (5) is a one-to-one correspondence. In fact, such a claim is prevented, among other things, by the circumstance that the assumptions placed by (iii) on f and g, assumptions consisting of (15) and of the mere continuity of f and g, are compatible with a variety of

solution paths of (5) which pass through the same point $(x_0, y_0) \neq (0,0)$; cf. the remarks made before (i).

What will actually be ascertained is an integral refinement of the order restriction (15):

(iii*) *The assertions of* (iii) *remain true if* (15) *is relaxed to*

$$(16) \qquad |f(x,y)| \leqq \psi(r) \qquad |g(x,y)| \leqq \psi(r),$$

where $\psi(r)$ *is any continuous, monotone non-decreasing function of* r *defined for* $0 \leqq r \leqq \alpha$ *in such a way that*

$$(17) \qquad \psi(r) = o(r) \qquad (r \to 0)$$

and

$$(18) \qquad \int_{+0}^{a} r^{-2}\psi(r)\,dr < \infty.$$

On the other hand, *some* restriction of the o in (2) is indispensable:

(iii bis) *The assertions of* (iii) *can fail if* (15) *is relaxed to* (2).

In order to see this, let, in virtue of (3),

$$f(x,y) = h(r)r\cos\theta, \qquad g(x,y) = h(r)r\sin\theta,$$

where $h(r)$ is a continuous function vanishing at $r = 0$. Then f and g are continuous functions satisfying (2). But (12) now becomes $r' = -r + rh(r)$. In particular, if $h(r) = 1/\log r$, then $w' = -1 + 1/w$, where $w = \log r$. This differential equation for w gives $w + \log(w - 1) = t_0 - t$, that is, $r(\log r - 1) = r_0 e^{-t}$. Hence, the assertion of (iii), which implies that $r(t) \sim r_0 e^{-t}$, where r_0 (> 0) is arbitrary, cannot be true in this case. This proves (iii bis).

4. In the proof of (iii) and of its generalization (iii*), it can be assumed that (4) and (5) are in their respective normal forms, (10) and (13).

Let u and v in (14) be thought of as functions of t, to be determined in such a way that (14) becomes a solution (6) of (10) (variation of constants). To this end, the formal substitution of (14) into (10) supplies the necessary and sufficient conditions

$$(19) \qquad u' = (f\cos\lambda t + g\sin\lambda t)e^{t}, \qquad v' = (f\sin\lambda t - g\cos\lambda t)e^{t},$$

where $f = f(x,y)$ and $g = g(x,y)$ must be expressed, in terms of (14), as

functions of u, v and t. In other words, (19) represents two differential equations of the form

$$(20) \qquad u' = f^*(u, v, t), \qquad v' = g^*(u, v, t),$$

where f^* and g^* are continuous functions defined in the region

$$(21) \qquad -\infty < t < \infty, \quad (u^2 + v^2)^{\frac{1}{2}} e^{-t} \leqq \alpha$$

of the (u, v, t)-space.

By the asymptotic correspondences referred to in (iii) and (iii*) is meant the following: If (6) is any solution of (10) satisfying $r(t) = o(1)$, as $t \rightarrow \infty$, then the corresponding solution

$$(22) \qquad u = u(t), \qquad v = v(t)$$

of (20) is such that the limits

$$(23) \qquad \lim_{t\to\infty} u(t) = u(\infty), \qquad \lim_{t\to\infty} v(t) = v(\infty)$$

exist; conversely, if a pair of constants $u(\infty)$, $v(\infty)$ is arbitrarily specified, then there exists at least one solution (22) of (20) satisfying (23).

According to (19), the functions f^*, g^* occurring in (20) are majorized by $(f^2 + g^2)^{\frac{1}{2}} e^t$. Hence, if the factor $2^{\frac{1}{2}}$ is thought of as being submerged into the function sign ψ, then, by (14) and (16),

$$|f^*(u, v, t)| \leqq \psi((u^2 + v^2)^{\frac{1}{2}} e^{-t}) e^t, \qquad |g^*(u, v, t)| \leqq \psi((u^2 + v^2)^{\frac{1}{2}} e^{-t}) e^t,$$

if (u, v, t) is a point in the region (21). Let C denote an arbitrary positive number and let $T = T(C)$ be defined by $Ce^{-T} = \alpha$. Then, by the monotony of ψ,

$$|f^*(u, v, t)| \leqq \psi(Ce^{-t}) e^t \qquad |g^*(u, v, t)| \leqq \psi(Ce^{-t}) e^t,$$

if (u, v, t) is a point in the region

$$(24) \qquad u^2 + v^2 \leqq C^2, \qquad T \leqq t < \infty.$$

Thus, it is possible to define a pair of functions $F(u, v, t)$, $G(u, v, t)$ which are continuous in the product space of the entire (u, v)-plane and the half-line $0 \leqq t < \infty$, are given by

$$(25) \qquad F(u, v, t) = f^*(u, v, t), \qquad G(u, v, t) = g^*(u, v, t)$$

when (u, v, t) is a point of the region (24), and satisfy the inequality

$$(26) \qquad (F(u, v, t)^2 + G(u, v, t)^2)^{\frac{1}{2}} \leqq \lambda(t)$$

for all points (u, v, t), where

$$\lambda(t) = \begin{cases} \psi(Ce^{-t})e^t \text{ if } Ce^{-t} \leqq \alpha, \\ \psi(\alpha)e^t \text{ if } Ce^{-t} > \alpha. \end{cases} \tag{27}$$

In order to apply to the system of differential equations

$$u' = F(u, v, t), \qquad v' = G(u, v, t) \tag{28}$$

the theorem proved in [3], let the majorant, $\lambda(t)$, on the right of (26) be written in the form $\lambda(t)\phi(u^2 + v^2)$, where $\phi(r) \equiv 1$. Then

$$\int_0^\infty \lambda(t)dt < \infty \text{ and } \int_1^\infty dr/\phi(r) = \infty, \tag{29}$$

since

$$\int_0^\infty \lambda(t)dt = \int_0^T \psi(\alpha)e^t dt + \int_T^\infty \psi(Ce^{-t})e^t dt$$

and

$$\int_T^\infty \psi(Ce^{-t})e^t dt = C\int_0^\alpha \psi(r)r^{-2}dr < \infty,$$

by (18).

It now follows from the general theorem of [3] that, if (22) is any solution of (28), then the limits (23) exist, and that the latter can attain arbitrary values $u(\infty)$, $v(\infty)$ when (22) is suitably chosen.

On the other hand, since (25) holds on the region (24), it is clear that, if (22) is any solution of (30) which satisfies

$$u(t)^2 + v(t)^2 \leqq C^2 \text{ for sufficiently large } t, \tag{30}$$

then it is a solution of (20) for all large t, and conversely. Since the constant C was arbitrary, it follows that the proof of (iii*) can be completed by showing that, if (6) is any solution of (10) satisfying $r(t) = o(1)$, then, for the corresponding solution (22) of (20), there exists a constant C satisfying (30). In view of (14), this is equivalent to the assertion

$$r(t)e^t = O(1), \qquad (t \to \infty). \tag{31}$$

If $r > 0$ is small enough, then, according to (12) and (16),

$$r' \leqq -r + 2\psi(r).$$

On the other hand, (17) shows that the inequality $r - 2\psi(r) > 0$ holds whenever $r > 0$ is small enough. Since $r = r(t)$ tends to 0 as $t \to \infty$, it follows that, when t is large enough,

$$(r - 2\psi(r))^{-1} r' < -1.$$

Finally, the identity

$$(r - 2\psi(r))^{-1} = r^{-1} + 2\psi(r)(r^2 - 2r\psi(r))^{-1},$$

when integrated with respect to t, supplies the inequality

$$r \leqq \text{const.}\, e^{-t} \exp\left(2 \int_r^{\cdot} \psi(u)(u^2 - 2u\psi(u))^{-1} du\right), \qquad r = r(t).$$

Clearly, (31) now follows from (17) and (18).

This completes the proof of (iii) and of its generalization (iii*).

5. The preceding proof has nowhere used the assumption that $\lambda \neq 0$ in (10), (13), (14). But if $\lambda = 0$, then (13) and (14) become $x' = -x$, $y' = -y$ and $x = x_0 e^{-t}$, $y = y_0 e^{-t}$ respectively, representing (except for the unit of length and the orientation on the t-axis) the only case in which $(0,0)$ is a proper node of (5); cf. the second section in **2** above. Correspondingly, it is clear from the definition of a proper node that the assertions of (iii) now become equivalent to the statement that $(0,0)$ is a proper node of (4). Accordingly, the proof of (iii) implies the following theorem:

(iv) *If* $(0,0)$ *is a proper node of* (5) *and if* (2) *in* (4) *is refined to* (15), *then* $(0,0)$ *is a proper node of* (4) *also.*

This limiting case of (iii) was proved by Perron ([2], p. 123, Fall II) under the assumption of continuous partial derivatives f_x, f_y, g_x, g_y for the functions $f(x,y)$, $g(x,y)$ in a circle about $(0,0)$. In (iv), these derivatives need not exist at any $(x,y) \neq (0,0)$, the functions f, g being just continuous. Correspondingly, Perron's additional statement that just one solution path of the system (4) issues from $(0,0)$ in every direction is not true under the general assumption of (iv), since the situation is the same as in the observation following (iii).

Perron's method of proof in his particular case of (iv) consists in a reduction of the system (4) to a single equation $dy/dx = F(x,y)$. Such a reduction is made possible, of course, only by the circumstance that $x = x(t)$ or $y = y(t)$ in (4) becomes substantially monotone near the node $(0,0)$. Since this circumstance is not presented by the spirals about a vortex $(0,0)$, Perron's method would not lead to (iii) even under the assumption of continuous partial derivatives for f, g (cf. Perron [2], pp. 280-283, where only (ii b) is proved for vortices).

(iv*) *The assertion of* (iv) *remains true if* (15) *is relaxed to* (16), (17), (18).

In fact, (iv*) relates to (iii*) in the same way as (iv) to (iii).

However, *some* Tauberian restriction of the o in (2) is necessary in order that the assertion of (iv) be correct. In fact, the truth of the negation which corresponds to (iii bis) is a corollary of (ii a).

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ANALYTICITY IN HILBERT SPACE AND SELF-ADJOINT TRANSFORMATIONS.*

By Martinus Esser.

1. Introduction. The spectral theory of self-adjoint transformations in Hilbert space has been developed in many different ways, both for particular cases and for the general case [von Neumann],[1] [Stone] etc. One method, which has formerly been used only for two particular types of self-adjoint transformations [Hellinger I and II], is based on Cauchy's integral theorem for analytic functions, and obtains the spectrum of the transformation by studying the singularities of certain analytic functions. The present paper will show how this method can be used to derive the whole theory of self-adjoint transformations. To this end, we must synthetize the usual features of analyticity and of Hilbert space, and use the notion of analytic dependency of elements in Hilbert space on a complex variable. We begin by studying integrals of elements in Hilbert space. Such integrals will be used for contour integrations of analytic elements and in the formulation of our final result [cf. Maeda and, later, Riesz and Lorch].

2. Integrals in Hilbert space. We consider a Hilbert space H. To distinguish between points $f(\lambda)$ in H and complex numbers $L(\lambda)$, both depending on a complex variable λ, we shall ascribe the word "element" to the first case and restrict the word "function" to the second case. The inner product of two elements f, g will be denoted by (f, g) and the modulus $(f, f)^{1/2}$ will be denoted by $|f|$. Definitions of equality, limits, continuity, series and integrals in the space H can be made in the usual manner by means of the modulus $|f|$. Two types of Stieltjes integrals will be considered, namely

$$\int_\Gamma f(\lambda)\,dL(\lambda) = \lim_{\delta\to 0} \sum_n f(\lambda'_n)\Delta_n L(\lambda) \tag{1}$$

and

$$\int_\Gamma L(\lambda)\,df(\lambda) = \lim_{\delta\to 0} \sum_n L(\lambda'_n)\Delta_n f(\lambda), \tag{2}$$

* Received November 9, 1946. The present article is derived from the first part of my doctoral dissertation [Esser]. The dissertation has been written under the guidance of Ernst Hellinger, Northwestern University.

[1] Names in brackets refer to references.

where $f(\lambda)$ is an element in H, $L(\lambda)$ a complex valued function and λ a complex variable describing a curve Γ. These integrals have properties similar to those of ordinary Stieltjes integrals, and we shall assume such properties without proof. In particular the integral (1) exists when the element $f(\lambda)$ is continuous and the variation of $L(\lambda)$ on Γ is finite. A similar sufficient condition of existence can be formulated for the integral (2) if we define the variation of an element as follows:

Definition I. The variation of an element $f(\lambda)$ on a curve Γ is defined to be the least upper bound of $|\sum_n a_n \Delta_n f(\lambda)|$ for all possible partitions of the curve Γ and for any complex numbers a_n whose modulus does not exceed one.

It should be noted that the analogies between the variation of functions and the variation of elements are not as complete as the analogies between integrals of functions and integrals of elements. For instance the variation of an element $f(\lambda)$ does not equal, in general, the least upper bound of $\Sigma |\Delta_n f(\lambda)|$, and the variation of an element over the sum of two intervals of λ may differ from the sum of the variations over each interval.

We have the following theorem:

THEOREM II. *If an element $f(x)$ depending on a real variable x satisfies the orthogonality relation*

$$(f(b)-f(a), f(d)-f(c)) = 0 \tag{3}$$

for any set of numbers $a \leq b \leq c \leq d$, then the variation of $f(x)$ on any interval (y, z) is finite and equals $|f(z)-f(y)|$.

Proof. We divide the interval (y, z) into successive intervals and denote by $\Delta_n f$ the increment of $f(x)$ over the n-th interval. We consider also complex numbers a_n with $|a_n| \leq 1$. We have then, by the linear properties of inner products,

$$(\sum_n a_n \Delta_n f, \sum_n a_n \Delta_n f) = \sum_{n,m} a_n \bar{a}_m (\Delta_n f, \Delta_m f). \tag{4}$$

By the orthogonality relation (3), and by the definition of moduli of elements, relation (4) becomes $|\sum_n a_n \Delta_n f|^2 = \sum_n |a_n|^2 |\Delta_n f|^2$. Considering different sets of a_n, we obtain the least upper bound of the second member by taking each a_n equal to one, and looking at the first member, we see that this upper bound is $|\Sigma \Delta_n f|^2 = |f(z)-f(y)|^2$. This value is independent of the subintervals of (y, z) considered, and therefore equals the square of the variation of $f(x)$ on that interval.

3. Analyticity in Hilbert space. A limit in H of the form

$$\frac{df(\lambda)}{d\lambda} = \lim_{\mu\to\lambda} \frac{f(\mu) - f(\lambda)}{\mu - \lambda},$$

where $f(\lambda)$ is an element in H and λ is a complex variable, will be called a derivative in H. An element $f(\lambda)$ which is defined and has a derivative for each λ of an open domain D in the λ plane, will be said to be analytic over D.

The usual properties of analytic functions can be generalized to analytic elements [Cf. Wiener]. We have the following theorems:

Let $f(\lambda)$ be an analytic element over a domain D. Then:

THEOREM III. *For any element g, the function $(f(\lambda), g)$ is analytic over D and has the derivative $(df(\lambda)/d\lambda, g)$.*

THEOREM IV. *For any two points a, b in D, the integral $\int_a^b f(\lambda)d\lambda$ exists and is independent of the curve of integration joining a to b, provided that such curves are not separated by points not belonging to D.*

THEOREM V. *Considering a curve Γ in D encircling once and in the positive sense a point λ, we have Cauchy's formula*

$$2\pi i\, f(\lambda) = \int_\Gamma (\mu - \lambda)^{-1} f(\mu)\, d\mu.$$

THEOREM VI. *For each point μ in D, we have the Taylor series expansion*

$$f(\lambda) = \sum_{n=0}^{\infty} (1/n!)\,(\lambda - \mu)^n\,(d^n f(\mu)/d\mu^n),$$

convergent over any circle centered at μ and contained in D.

Only Theorems III and IV will be used in this article. The integral in Theorem IV exists for each curve joining a to b because $f(\lambda)$ is continuous, and the integral is independent of the path because, for each element g, the number $(\int_a^b f(\lambda)d\lambda, g) = \int_a^b (f(\lambda), g)\,d\lambda$ is independent of the path.

4. Self-adjoint transformations. The analytic element with which we shall be concerned in this article is the element whose existence is stated by the following theorem.

THEOREM VII. *Considering a self-adjoint transformation T, an arbitrary element u in H, and a non-real complex variable λ, the relation*

(5) $$T[f(\lambda)] - \lambda f(\lambda) = u$$

defines a unique element $f(\lambda)$. *This element is analytic and its derivative satisfies the relation*

(6) $$T[df(\lambda)/d\lambda] - \lambda df(\lambda)/d\lambda = f(\lambda).$$

Denoting by λ_2 *the imaginary part of* λ, *we have the inequality*

(7) $$|f(\lambda)| \leq |\lambda_2|^{-1} |u|.$$

The existence and uniqueness of $f(\lambda)$, and the inequality (7) are well-known [Stone, Definition 2.11, Theorem 4.14]. The existence of the derivative can be shown as follows: The transformation T being linear, we have

(8) $$T[f(\lambda) - f(\mu)] - \lambda[f(\lambda) - f(\mu)] = (\lambda - \mu)f(\mu).$$

Inequality (7) gives then

$$|f(\lambda) - f(\mu)| \leq |\lambda_2|^{-1} |\lambda - \mu| \, |f(\mu)| \leq |\lambda_2 \mu_2|^{-1} |\lambda - \mu| \, |u|.$$

The right member of this inequality approaches zero when μ approaches λ, therefore $f(\lambda)$ is continuous.

We divide both members of (8) by $(\lambda - \mu)$:

(9) $$T\left[\frac{f(\lambda) - f(\mu)}{\lambda - \mu}\right] - \lambda \frac{f(\lambda) - f(\mu)}{\lambda - \mu} = f(\mu),$$

and let μ approach λ. The second member approaches $f(\lambda)$. In the first member $[f(\lambda) - f(\mu)]/[\lambda - \mu]$ must then approach a limit, because the transformation from $f(\mu)$ to $[f(\lambda) - f(\mu)]/[\lambda - \mu]$ defined by equation (9) is a closed bounded transformation [Stone, Definitions 2.5 and 2.13]. Therefore $df(\lambda)/d\lambda$ exists and satisfies relation (6). Theorem VII is thus proved.

We shall now find a few properties of the element $f(\lambda)$ which will justify the consideration of Lemma VIII below.

We consider elements satisfying the two relations

(10) $$\begin{aligned} T[f(\lambda)] - \lambda f(\lambda) &= u, \\ T[g(\mu)] - \mu g(\mu) &= v. \end{aligned}$$

By definition of self-adjoint transformations, we have $(T[f(\lambda)], g(\mu)) = (f(\lambda), T[g(\mu)])$. Substituting for $T[f]$ and $T[g]$ the values given by (10), and using the linear properties of inner products, we get

(11) $$(\lambda - \bar{\mu})(f(\lambda), g(\mu)) = (f(\lambda), v) - (u, g(\mu)),$$

where $\bar{\mu}$ denotes the conjugate of μ.

We introduce a function $L(\lambda)$ defined by

$$L(\lambda) = (f(\lambda), u) = \overline{(u, f(\lambda))}. \tag{12}$$

By Schwarz's inequality and inequality (7), we have

$$|L(\lambda)| \leq |f(\lambda)|\ |u| \leq |\lambda_2|^{-1} |u|^2. \tag{13}$$

By taking $v = u$, $g(\mu) = f(\mu)$ in equation (11) and using equation (12), we obtain

$$(\lambda - \mu)(f(\lambda), f(\mu)) = L(\lambda) - \overline{L(\mu)}. \tag{14}$$

5. Proof of a lemma. Relations (13) and (14) lead to a lemma whose proof constitutes the essential part of our construction of the spectrum of the transformation T.

LEMMA VIII. *Let $f(\lambda)$ be an analytic element defined for non-real λ, and satisfying for each non-real λ and μ the relation*

$$(\lambda - \mu)(f(\lambda), f(\mu)) = L(\lambda) - \overline{L(\mu)}, \tag{15}$$

where $L(\lambda)$ is a function for which $|\lambda_2 L(\lambda)|$ stays bounded. Then there exists an element $u(x)$ depending on the real variable x, which has a finite variation on the interval $-\infty \leq x \leq +\infty$, and such that

$$f(\lambda) = \int_{-\infty}^{+\infty} du(x)/(x - \lambda). \tag{16}$$

The element $u(x)$ is continuous on the right

$$u(x) = \lim_{\epsilon \to 0} u(x + \epsilon), \qquad \epsilon > 0, \tag{17}$$

and satisfies the orthogonality relation

$$(u(b) - u(a),\ u(d) - u(c)) = 0 \tag{18}$$

for any set of numbers $a \leq b \leq c \leq d$.

The function $L(\lambda)$ has the three following properties:

1). $L(\lambda)$ is analytic in the upper half λ plane, as is seen by Theorem III applied to equation (15).

2). $L(\lambda)$ has a non-negative imaginary part $L_2(\lambda)$ for positive λ_2, because, for $\lambda = \mu$, relation (15) becomes $\lambda_2 |f(\lambda)|^2 = L_2(\lambda)$.

3). By hypothesis, $\lambda_2 |L(\lambda)|$ is bounded.

It is known [Doob and Koopman, Esser] that functions $L(\lambda)$ which have the three preceding properties can be represented, for $\lambda_2 > 0$, by the integral

$$(19) \qquad L(\lambda) = \int_{-\infty}^{+\infty} (x-\lambda)^{-1} d\rho(x),$$

where $\rho(x)$ is a certain real, bounded, non-decreasing function. We may suppose $\rho(x)$ continuous on the right.

Moreover, when we make $\lambda = \bar{\mu}$ in equation (15), we see that $\overline{L(\lambda)} = L(\bar{\lambda})$, and therefore relation (19) is also valid when λ is in the lower half plane. Substituting the integral (19) for L in equation (15), we obtain successively

$$(20) \qquad (\lambda - \bar{\mu})(f(\lambda), f(\mu)) = \int_{-\infty}^{+\infty} \frac{d\rho(x)}{x-\lambda} - \int_{-\infty}^{+\infty} \frac{d\rho(x)}{x-\bar{\mu}},$$

$$(21) \qquad (f(\lambda), f(\mu)) = \int_{-\infty}^{+\infty} \frac{d\rho(x)}{(x-\lambda)(x-\bar{\mu})}.$$

This formula shows us how the inner product $(f(\lambda), f(\mu))$ depends on the variables λ, μ, and thereby will enable us to study convergence properties of the element $f(\lambda)$.

We consider two points a, b in the same half λ plane, and integrate the two members of (21) with respect to λ over a curve from a to b in that half λ plane. In the first member, the integral of the inner product may be replaced by the inner product of the integral of $f(\lambda)$ by $f(\mu)$. In the second member, the order of the two integrations can be interchanged because of the uniform convergence of the Stieltjes integral. We thus obtain

$$\left(\int_a^b f(\lambda)\, d\lambda, f(\mu)\right) = \int_{-\infty}^{+\infty} \int_a^b \frac{d\lambda}{x-\lambda} \frac{d\rho(x)}{x-\bar{\mu}}.$$

We rewrite this equation with a and b replaced by $\bar{b}$ and $\bar{a}$, and add the two equations. Defining, except for an additive element independent of λ, an element $u(\lambda)$ by

$$(22) \qquad 2\pi i[u(b) - u(a)] = \int_a^b f(\lambda)\, d\lambda + \int_{\bar{b}}^{\bar{a}} f(\lambda)\, d\lambda,$$

we obtain

$$(23) \qquad \pi(u(b) - u(a), f(\mu)) = \int_{-\infty}^{+\infty} \arg \frac{a-x}{b-x} \frac{d\rho(x)}{x-\bar{\mu}},$$

where arg stands for the argument contained between $-\pi$ and $+\pi$.

By repeating the sequence of operations which transformed (21) into (23), we transform (23) into

$$\pi^2(u(b)-u(a),u(d)-u(c))=\int_{-\infty}^{+\infty}\arg\frac{a-x}{b-x}\arg\frac{c-x}{d-x}\,d\rho(x), \tag{24}$$

where a, b, c, d are any four numbers with positive imaginary parts. If, in particular, we take $c=a$, $d=b$, we obtain

$$\pi^2\,|\,u(b)-u(a)\,|^2=\int_{-\infty}^{+\infty}\arg^2\frac{a-x}{b-x}\,d\rho(x). \tag{25}$$

The function $\rho(x)$, being monotone, is continuous for almost all x. When a and b approach a real point y where $\rho(x)$ is continuous, then the integral in (25) approaches zero. Therefore $u(a)$ converges when a approaches y, and we can define an element $u(y)$ by $u(y)=\lim\limits_{a\to y}u(a)$. If we let a approach a real point y of continuity of $\rho(x)$, the members of equations (23), (24) and (25) approach limits, which are obtained by replacing a by y in these members. Similarly, we can let b, c, d approach real points where $\rho(x)$ is continuous, and thus find that the equations (23), (24), (25) stay valid when a, b, c, d are real points where $\rho(x)$ is continuous. The formulas so obtained are:

$$(u(b)-u(a),f(\mu))=\int_a^b\frac{d\rho(x)}{x-\mu},\qquad a\leq b, \tag{26}$$

$$(u(b)-u(a),u(d)-u(c))=0,\qquad a\leq b\leq c\leq d, \tag{27}$$

$$|\,u(b)-u(a)\,|^2=\rho(b)-\rho(a),\qquad a\leq b. \tag{28}$$

When a and b approach from the right an arbitrary real point x, the members of (28) approach zero. Therefore $\lim\limits_{\epsilon\to 0}u(x+\epsilon)$, where $x+\epsilon$ varies over the points of continuity of $\rho(x)$ on the right of x, exists for all x. This limit will define $u(x)$ at the points of discontinuity of $\rho(x)$.

Thus we have defined for all real x an element $u(x)$, which will be the element $u(x)$ mentioned in Lemma VIII. This element satisfies the relation (17). Moreover, by letting one or more of the points a, b, c, d approach, from the right, points of discontinuity of $\rho(x)$, we see that formulas (26), (27) and (28) remain valid when one or more of the points a, b, c, d are points of discontinuity of $\rho(x)$. Thus in particular the orthogonality relation (18) is proved.

Since $\rho(x)$ is bounded and monotone, the limits $\rho(-\infty)=\lim\limits_{x\to-\infty}\rho(x)$ and $\rho(+\infty)=\lim\limits_{x\to+\infty}\rho(x)$ exist. Therefore, by (28), the limits $u(-\infty)=\lim\limits_{x\to-\infty}u(x)$ and $u(+\infty)=\lim\limits_{x\to+\infty}u(x)$ exist. The element $u(x)$ has been determined hitherto except for an additional element independent of x. This element may be determined by the additional condition

$$(29) \qquad u(-\infty) = \lim_{x \to -\infty} u(x) = 0.$$

By Theorem II, the variation of $u(x)$ on the infinite interval of x is finite. It equals $[\rho(+\infty) - \rho(-\infty)]^{\frac{1}{2}}$.

The integral of equation (16) exists because $(x-\lambda)^{-1}$ is continuous and the variation of $u(x)$ is finite. By the construction used in its definition, the integral is contained in the closed linear manifold determined by the values of the element $f(\mu)$ considered for all non-real μ. Therefore, to prove equation (16), it is sufficient to show that its two members have equal projections on each element $f(\mu)$, in other words that for all non-real μ we have

$$(30) \qquad (f(\lambda), f(\mu)) = \left(\int_{-\infty}^{+\infty} \frac{du(x)}{x-\lambda}, f(\mu)\right).$$

Using (26) and (21), we get successively

$$\left(\int_{-\infty}^{+\infty} \frac{du(x)}{x-\lambda}, f(\mu)\right) = \int_{-\infty}^{+\infty} \frac{1}{x-\lambda}\, d(u(x), f(\mu))$$
$$= \int_{-\infty}^{+\infty} \frac{1}{x-\lambda}\, d\int_{-\infty}^{x} \frac{d\rho(y)}{y-\bar{\mu}} = \int_{-\infty}^{+\infty} \frac{d\rho(x)}{(x-\lambda)(x-\bar{\mu})} = (f(\lambda), f(\mu)),$$

which proves (30) and therefore (16). The proof of Lemma VIII is thus completed.

6. Spectral projections. Given a self-adjoint transformation T, the correspondence of elements $f(\lambda)$ to elements u established by relation (5), and of elements $u(x)$ to elements $f(\lambda)$ established in Section **5**, associates with each element u in H an element $u(x)$ defined and unique for each real x. This element $u(x)$ will be called the "spectral projection" of u. In the present section, we shall show mainly that for each real x, $u(x)$ is the projection of u on a linear manifold $\mathfrak{M}_x$, this manifold depending only on x and T, but not on u.

We shall first show that

$$(31) \qquad u(+\infty) = \lim_{x \to +\infty} u(x) = u.$$

Equations (26), (29), (19) and (12) give successively

$$(u(+\infty), f(\mu)) = \int_{-\infty}^{+\infty} \frac{d\rho(x)}{x-\bar{\mu}} = \overline{L(\mu)} = (u, f(\mu)).$$

Therefore, and because $u(+\infty)$ is in the closed linear manifold determined by $f(\mu)$ for all non-real μ, we find that $u(+\infty)$ is the projection of u on

this manifold. To establish (31), we have yet to show that u is in this manifold.

If u is in the domain $\mathfrak{D}$ of T, we get successively

$$T\{T[f(\lambda)]\} - \lambda T[f(\lambda)] = T[u] \qquad \text{by (5)},$$

$$|T[f(\lambda)]| \leq |\lambda_2|^{-1} |T[u]| \qquad \text{by (7)},$$

$$\lim_{\lambda_2 \to +\infty} T[f(\lambda)] = 0,$$

$$u = \lim_{\lambda_2 \to +\infty} \lambda f(\lambda) \qquad \text{by (5)}.$$

The last equation shows that u is in the closed linear manifold determined by $f(\lambda)$ for all non-real λ, and therefore $u = u(+\infty)$. This equality can be extended to arbitrary elements u in H because the domain $\mathfrak{D}$ of T is everywhere dense in H and the transformation from u to $u(+\infty)$ is linear and bounded. Therefore (31) is proved.

We show next that the orthogonality relation (18) can be generalized into

$$(32) \qquad (u(b) - u(a), v(d) - v(c)) = 0, \qquad a \leq b \leq c \leq d,$$

where $u(x)$, $v(x)$ are the spectral projections of two different elements u, v. To these elements u, v correspond elements $f(\lambda)$, $g(\mu)$ defined by equations (10). By (16) we get

$$(33) \qquad (f(\lambda), v) = \int_{-\infty}^{+\infty} \frac{d\sigma(x)}{x - \lambda},$$

where $\sigma(x) = (u(x), v)$. The function $\sigma(x)$ has a finite variation on the infinite interval because the variation of the element $u(x)$ is finite, and it is continuous on the right because $u(x)$ is continuous on the right. By taking $\lambda = \bar{\mu}$ in equation (11), we see that

$$(u, g(\mu)) = (f(\bar{\mu}), v) = \int_{-\infty}^{+\infty} \frac{d\sigma(x)}{x - \bar{\mu}}.$$

Therefore equation (11) becomes

$$(\lambda - \bar{\mu})(f(\lambda), g(\mu)) = \int_{-\infty}^{+\infty} \frac{d\sigma(x)}{x - \lambda} - \int_{-\infty}^{+\infty} \frac{d\sigma(x)}{x - \bar{\mu}}.$$

From this equation we can derive (32) by the method used in deriving (18) from (20).

By particularizing a, b, c, d in (32), and using (29) and (31), we get $(u(x), v - v(x)) = 0$ for each x. Thus, for each x, the linear manifold $\mathfrak{M}_x$ formed by all elements $u(x)$ is orthogonal to the linear manifold formed

by all elements $u - u(x)$. As each element u in H is the sum of an element $u(x)$ in $\mathfrak{M}_x$ and an element orthogonal to $\mathfrak{M}_x$, the manifold $\mathfrak{M}_x$ must be closed and $u(x)$ is the projection of u on $\mathfrak{M}_x$.

The orthogonality relation (32) gives, moreover, $(u(x), v - v(y)) = 0$ for $x \leq y$. Therefore $u(x)$ belongs to the manifold $\mathfrak{M}_y$ of all elements orthogonal to all $v - v(y)$, and we infer that $\mathfrak{M}_x$ is contained in $\mathfrak{M}_y$.

7. The transformation T. This section will prove that an element f, with spectral projection $f(x)$, is in the domain $\mathfrak{D}$ of the transformation T if, and only if, the integral $\int_{-\infty}^{+\infty} x\, df(x)$ exists, and that we have then

$$T[f] = \int_{-\infty}^{+\infty} x\, df(x). \tag{34}$$

Let us first suppose that f is an element in $\mathfrak{D}$. We define an element u by

$$u = T[f] - if. \tag{35}$$

Then formula (16) gives $f = \int_{-\infty}^{+\infty} (y - i)^{-1} du(y)$. Projecting on $\mathfrak{M}_x$, we get $f(x) = \int_{-\infty}^{x} (y - i)^{-1} du(y)$. From this integral we obtain, for each real a and b, $u(b) - u(a) = \int_a^b (x - i)\, df(x)$. The first member has the limit u when $a \to -\infty$, $b \to +\infty$. Therefore $\int_{-\infty}^{+\infty} (x - i)\, df(x)$ exists and we have

$$u = \int_{-\infty}^{+\infty} (x - i)\, df(x) = \int_{-\infty}^{+\infty} x\, df(x) - if. \tag{36}$$

Comparing (35) and (36), we obtain (34).

Conversely, let f be such that the integral in (34) exists. Then we define u by (36). We obtain $u(x) = \int_{-\infty}^{x} (y - i)\, df(y)$. The integral $\int_{-\infty}^{+\infty} (x - i)^{-1} du(x)$ exists and equals f. Therefore f is the element which satisfies (35). It results that f is in $\mathfrak{D}$ and satisfies again (34).

8. Conclusion. We have shown how the theory of analytic functions can be used to derive the known characterization of self-adjoint transformations, which is:

THEOREM IX. *Each self-adjoint transformation T in Hilbert space can be defined as follows by means of a certain family of closed linear manifolds*

$\mathfrak{M}_x$: *Let* $f(x)$ *denote the projection of* f *on* $\mathfrak{M}_x$. *Then* f *is in the domain of* T *if and only if the integral* $\int_{-\infty}^{+\infty} x\, df(x)$ *exists. The integral equals then* $T[f]$.

The manifolds $\mathfrak{M}_x$ *are such that:*

1). *For* $x \leq y$, $\mathfrak{M}_x$ *is a subset of* $\mathfrak{M}_y$.

2). *For each* f *we have* $\lim_{x\to-\infty} f(x) = 0$ *and* $\lim_{x\to+\infty} f(x) = f$.

3). *The element* $f(x)$ *is continuous on the right*: $f(x) = \lim_{\epsilon\to 0} f(x+\epsilon)$, $\epsilon > 0$.

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ON THE SUMMABILITY OF MULTIPLE FOURIER SERIES.*

By A. Zygmund.

1. Let $f(x,y)$ be an L-integrable function of period 2π with respect to both x and y. Let

$$\sum_{m,n=-\infty}^{+\infty} c_{mn} e^{i(mx+ny)} \tag{1}$$

be the Fourier series of f, so that

$$c_{mn} = 1/(2\pi)^2 \int_0^{2\pi}\int_0^{2\pi} f(x,y) e^{-i(mx+ny)}\,dxdy$$

for all m and n. As regards the convergence of the first arithmetic means

$$\begin{aligned}\sigma_{m,n}(x,y) &= \sigma_{m,n}(x,y;f)\\ &= \sum_{\mu,\nu=-m,-n}^{m,n} (1-(|\mu|/(m+1))(1-(|\nu|/(n+1))e^{i(\mu x+\nu y)}\end{aligned} \tag{2}$$

of the series (1) the following results are known.

Theorem A. *There is an f such that $\sigma_{mn}(x,y)$ diverges everywhere. More precisely,*

$$\limsup_{m,n\to+\infty} \sigma_{m,n}(x,y) = +\infty$$

everywhere.

Theorem B. *If m and n tend to $+\infty$ in such a way that the ratios m/n and n/m remain bounded, then*

$$\sigma_{m,n}(x,y;f) \to f(x,y)$$

almost everywhere.

Theorem C. *If not only f but also $|f|\log^+|f|$ is integrable, then $\sigma_{m,n}(x,y;f)\to f(x,y)$ almost everywhere as m and n tend to $+\infty$ independently of each other.*

Corresponding results hold for the Abel means

$$f_{r,\rho}(x,y) = \sum_{m,n=-\infty}^{+\infty} c_{mn} e^{i(mx+ny)} r^{|m|}\rho^{|n|}$$

* Received February 1, 1947.

of (1) as $r, \rho \to 1$. The condition of boundedness of the ratios m/n and n/m is then to be replaced by the boundedness of the ratios $(1-r)/(1-\rho)$ and $(1-\rho)/(1-r)$.

All this, and the literature, will be found in Saks [5], Zygmund [7], Jessen, Marcinkiewicz and Zygmund [2], and Marcinkiewicz and Zygmund [3]. It may be added that considering the limit of σ_{mn} when m and n tend to infinity in such a way that m/n and n/m are bounded, was first suggested by C. N. Moore [4, p. 567]. That the last restriction is essential, follows from Theorem A. Of course, this kind of summability—let us call it *restricted summability* (C, 1)—is analogous to the classical concept of *restricted differentiability* of multiple integrals. For, as shown by Lebesgue, for any integrable $f(x, y)$,

$$\lim_{h,k\to 0} 1/hk \int_0^h \int_0^k f(x+u, y+v)\,du\,dv = f(x, y)$$

almost everywhere, provided h and k tend to 0 in such a way that the ratios h/k and k/h are bounded.

The main purpose of this note is to prove the following extension of Theorem B.

THEOREM 1. *Let $m = m(t)$ and $n = n(t)$ be any two non-decreasing and integer valued functions of the parameter t, $0 \leq t < \infty$, tending to infinity with t. Then, for any integrable $f(x, y)$*

$$\lim_{t\to+\infty} \sigma_{m,n}(x, y; f) = f(x, y)$$

at almost every point (x, y). More generally, let λ be any number ≥ 1. Then

$$\sigma_{\mu,\nu}(x, y; f) \to f(x, y)$$

almost everywhere, if μ and ν tend to infinity in such a way that

$$\lambda^{-1}m(t) \leq \mu \leq \lambda m(t), \qquad \lambda^{-1}n(t) \leq \nu \leq \lambda n(t). \tag{3}$$

The second part of this theorem reduces to the first part if $\lambda = 1$. If $m(t) = n(t) = [t]$ (= the integer part of t), we obtain Theorem B. For clearly, if Theorem 1 is proved for any fixed number $\lambda \geq 1$, it is automatically established for $\lambda = \lambda(x, y)$ varying from point to point. The novelty of Theorem 1, in comparison with Theorem B, is that m and n need no longer be of the same order of magnitude (consider, for example σ_{n,n^2} or $\sigma_{n,2^n}$). Their rate of increase may be totally different, provided that—

except for a bounded factor—one of them is a monotone function of the other.

The analogue of (the second part of) Theorem 1 for Abel means is as follows.

THEOREM 2. *Let $\phi(u)$ and $\psi(u)$ be non-decreasing functions of u defined in the interval $0 < u \leq 1$, satisfying the inequalities $0 < \phi(u) \leq 1$, $0 < \psi(u) \leq 1$, and tending to 0 with u. Let $\lambda \geq 1$ be fixed. Then, for every integrable f and almost every point (x, y),*

$$f_{r,\rho}(x, y) \rightarrow f(x, y)$$

as r and ρ tend to 1 in such a way that

$$\begin{aligned} &\lambda^{-1}\phi(u) \leq 1 - r \leq \lambda\phi(u), \\ &\lambda^{-1}\psi(u) \leq 1 - \rho \leq \lambda\psi(u). \end{aligned} \tag{4}$$

More generally, at almost every point (x, y) we have

$$f_{r,\rho}(\xi, \eta) \rightarrow f(x, y)$$

provided the points $re^{i\xi}$ and $\rho e^{i\eta}$ tend respectively to e^{ix} and e^{iy} along any non-tangential paths and in such a way that conditions (4) are satisfied.

It will be sufficient here to prove Theorem 1 only. The proof of Theorem 2 is completely analogous since the Poisson and the Féjér kernels satisfy similar inequalities (see inequalities (4) below), which are their only properties required in the proof. The fact that r and ρ are, unlike m and n, continuous variables does not affect the argument. Nor does the case of non-tangential paths mentioned in Theorem 2 introduce any new difficulty. For all that, Theorem 2 is more interesting in applications than Theorem 1. We shall return to some of these applications in another note.

The proof of Theorem 1 is given in Section **2**, below. Section **3** will be devoted to some additional results.

2. LEMMA 1. *Let $h(t)$ and $k(t)$ be two positive functions defined for $t > 0$, non-decreasing, and tending to 0 with t. Let E be any plane set whose outer measure $|E|$ is finite and positive. Suppose that to every point $(x, y) \in E$ corresponds a rectangle $R = R_{x,y}$ with center (x, y), and sides $2h(t)$, $2k(t)$ parallel to the axes, where $t = t(R)$ varies with R. Then there is a finite number of rectangles $R_{x_0y_0}, R_{x_1y_1}, \cdots, R_{x_ny_n}$ without points in common and such that*

$$\sum_{m=0}^{n} | R_m | > | E | /26 \tag{5}$$

where $R_m = R_{x_m y_m}$.

Proof. Let K_0 denote the aggregate of all the rectangles R corresponding to the points of E. Let

$$t^*_0 = \operatorname*{Sup}_{R \epsilon K_0} t(R). \tag{6}$$

We may assume that the sides of the rectangle $R \epsilon K_0$ are bounded. For otherwise there would exist rectangles R with areas arbitrarily large, and (5) could obviously be satisfied with $n = 0$.

Let us now define a rectangle R_0 and a number $t_0 = t(R_0)$ by the following conditions. If t^*_0 in (6) is actually attained, that is if there is a rectangle $R \epsilon K_0$ such that $t^*_0 = t(R)$, we take that R for R_0, and set $t_0 = t^*_0$. Otherwise, we take for R_0 any R such that $t_0 = t(R)$ satisfies both conditions

$$h(t_0) \geq \tfrac{1}{2} h(t^*_0 - 0), \qquad k(t_0) \geq \tfrac{1}{2} k(t^*_0 - 0).$$

Let us now denote by K'_1 the set of all the rectangles $R \epsilon K_0$ which have points in common with R_0, and let K_1 be the class of the remaining rectangles R. Thus $K_0 = K'_1 + K_1$. Moreover, it is immediately seen that if we denote by $\bar{R}_0$ the rectangle concentric with R_0, with sides parallel to the axes and dimensions five times those of R_0, then all the rectangles $R \epsilon K'_1$ are covered by $\bar{R}_0$.

Let us now set

$$t^*_1 = \operatorname*{Sup}_{R \epsilon K_1} t(R), \tag{7}$$

and let us define a rectangle R_1 and a number $t_1 = t(R_1)$ by the following conditions. If t_1 in (7) is attained, we take $t_1 = t^*_1$, and for R_1 we take the corresponding R. Otherwise, we take for R_1 any $R \epsilon K_1$ such that $t_1 = t(R)$ satisfies the conditions

$$h(t_1) \geq \tfrac{1}{2} h(t^*_1 - 0), \qquad k(t_1) \geq \tfrac{1}{2} k(t^*_1 - 0).$$

Thus R_1 has no points in common with R_0. Let K'_2 be the set of all the rectangles $R \epsilon K_1$ which have points in common with R_1, and K_2 the set of the remaining rectangles from K_1. Hence $K_1 = K'_2 + K_2$. Again, all rectangles from K'_2 are covered by the rectangle $\bar{R}_1$ concentric with and similar to R_1, with sides five times larger.

The general procedure is now clear. Suppose we have already defined

t^*_{m-1}, t_{m-1}, K_{m-1}, R_{m-1}. We then set $K_{m-1} = K'_m + K_m$, where K'_m consists of the rectangles $R \epsilon K_{m-1}$ which have points in common with R_{m-1}, and K_m of the remaining rectangles from K_{m-1}. The rectangles from K'_{m-1} are contained in the rectangle $\bar{R}_{m-1}$ concentric with, similar to, and five times larger than R_{m-1}. We set

$$(8) \qquad t^*_m = \operatorname*{Sup}_{t \epsilon K_m} t(R)$$

and set $t_m = t^*_m$ if the supremum in (8) is attained. Otherwise we take $t_m = t(R)$ for an $R \epsilon K_m$ and satisfying

$$h(t_m) \geq \tfrac{1}{2}h(t^*_m - 0), \qquad k(t_m) \geq \tfrac{1}{2}k(t^*_m - 0).$$

Obviously R_m has no point in common with $R_{m-1}, \cdots, R_1, R_0$.

The sequence $R_0, R_1, R_2, \cdots$ may be finite or not. In the former case, K_m is empty for some m. Let us first suppose that the sequence is infinite. Since $t^*_0 \geq t_1 \geq t_2 \geq \cdots$, there are two possibilities

(i) all the numbers t^*_m are bounded below by a positive number;

(ii) the numbers t^*_m tend to 0.

In case (i), inequality (5) is obvious for n large enough. Let us therefore pass to case (ii). It is easy to see that every rectangle R' from K_0 is contained in some $\bar{R}_m$. For suppose that this is not true. That would mean that for each m the rectangle R' is contained in K_m, which is clearly impossible since the dimensions of the rectangles from K_m do not exceed $2h(t^*_m)$, $2k(t^*_m)$, and so tend to 0 with $1/m$.

Since E is contained in the sum of the rectangles R from K_0, it must be contained in $\bar{R}_0 + \bar{R}_1 + \cdots$. Hence,

$$(9) \qquad | E | \leqq \sum_{m=0}^{\infty} | \bar{R}_m | = 25 \sum_{m=0}^{\infty} | R_m |,$$

and this gives (5) for n large enough.

If the sequence $\{R_m\}$ is finite and ends with R_n, the above argument gives (9) with ∞ replaced by n. The inequality (5) is then true *a fortiori*.

Remark. The above proof is a simple adaptation of the proof in the special case $h(t) = t, k(t) = \alpha t$, α being any positive but fixed number (see Marcinkiewicz and Zygmund [3]). It is easy to see that the coefficient 1/26 in (5) could be replaced by any number $> 1/9$, but the numerical value of

it is without importance. There is an extension of Lemma 1 to the case when each point of E belongs to infinitely many rectangles R with dimensions tending to 0 (see Jessen, Marcinkiewicz and Zygmund [2]). This extension is not needed here.

LEMMA 2. *Let $f(x,y)$ be an integrable function defined in the square*

$$(Q') \qquad -2\pi \leq x \leq 2\pi, \qquad -2\pi \leq y \leq 2\pi,$$

and let $h(t)$ and $k(t)$ be the functions of Lemma 1. For (x,y) belonging to the square

$$(Q) \qquad -\pi \leq x \leq \pi, \qquad -\pi \leq y \leq \pi$$

let

$$(10) \qquad f_*(x,y) = \operatorname*{Sup}_t 1/4hk \int_{-h}^{h} \int_{-k}^{k} | f(x+u, y+v) | \, dudv,$$

where t is so small that the rectangle over which we integrate is contained in Q'. For any $\xi > 0$, let $\mathcal{E}_*(\xi)$ denote the set of points $(x,y) \epsilon Q$ at which $f_*(x,y) > \xi$. Then

$$(11) \qquad | \mathcal{E}_*(\xi) | \leq 26\xi^{-1} \int\int_{Q'} | f(x,y) | \, dxdy.$$

Proof. If $(x,y) \epsilon \mathcal{E}_*(\xi)$, there is a rectangle R with center (x,y), with sides parallel to the axes and of length $2h(t)$, $2k(t)$. By Lemma 1, we can select a finite number of these rectangles without points in common and such that the set $E = \mathcal{E}_*(\xi)$ satisfies (5). This gives

$$\int\int_{Q'} | f | \, dxdy \geq \sum_{m=0}^{n} \int\int_{R_m} | f | \, dxdy > \xi \, | \mathcal{E}_*(\xi) | /26,$$

from which (11) follows.

LEMMA 3. *Let $h(t)$, $k(t)$, $f(x,y)$ be the functions of Lemma 2, let α and β be fixed positive numbers, and let $f_*^{\alpha,\beta}(x,y)$ be the functions $f_*(x,y)$ of Lemma 2 with $h(t)$, $k(t)$ replaced by $\alpha h(t)$ and $\beta k(t)$ respectively. For $(x,y) \epsilon Q$, let*

$$f^*(x,y) = \operatorname*{Sup}_{i,j} \{ f_*^{2^i,2^j}(x,y) 2^{-\frac{1}{2}(i+j)} \} \quad for \ i,j = 0,1,2,\cdots$$

and let $\mathcal{E}^*(\xi)$ be the set of points $(x,y) \epsilon Q$ at which $f^*(x,y) > \xi$. Then

$$(12) \qquad | \mathcal{E}^*(\xi) | \leq A\xi^{-1} \int\int_{Q'} | f | \, dxdy.$$

Proof. Let $\mathcal{E}_*^{\alpha,\beta}(\xi)$ be the set $\mathcal{E}_*(\xi)$ of Lemma 2 when we replace there $h(t)$, $k(t)$ by $\alpha h(t)$, $\beta k(t)$. A necessary and sufficient condition for the inequality $f^*(x,y) > \xi$ is that $f_*^{2^i,2^j}(x,y) > \xi 2^{(i+j)/2}$ for some non-negative integers i, j. Thus

$$\mathcal{E}^*(\xi) \subset \sum_{i,j=0}^{\infty} \mathcal{E}_*^{2^i,2^j}(\xi 2^{\frac{1}{2}(i+j)}),$$

and so

$$|\mathcal{E}^*(\xi)| \leqq \sum_{i,j=0}^{\infty} |\mathcal{E}_*^{2^i,2^j}(\xi 2^{\frac{1}{2}(i+j)})| \leq 26\xi^{-1}\Big(\sum_{i,j=0}^{\infty} 2^{-\frac{1}{2}(i+j)}\Big) \int\int_{Q'} |f|\, dxdy,$$

which leads to (12).

The arithmetic means $\sigma_{\mu\nu}$ of the Fourier series of f are given by the formula

$$(13) \qquad \sigma_{\mu\nu}(x,y) = \pi^{-2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+u, y+v) K_\mu(u)\, K_\nu(v)\, dudv$$

where $K_\mu(u)$ is the Féjér kernel,

$$K_\mu(u) = \frac{1}{\mu+1} \frac{\sin^2 \frac{1}{2}(\mu+1)u}{2 \sin^2 \frac{1}{2}u}$$

satisfying the inequalities

$$(14) \qquad K_\mu(u) \leq A\mu, \qquad K_\mu(u) \leq A\mu^{-1}u^{-2}$$

for $\mu \geq 1$ and $0 \leq u \leq \pi$. By A we mean here and hereafter a positive absolute constant not necessarily always the same.

Let

$$(15) \qquad \bar{\sigma}_{\mu\nu}(x,y) = \pi^{-2} \int_0^{\pi}\int_0^{\pi} |f(x+u, y+v)|\, K_\mu(u) K_\nu(v)\, dudv$$

and let $m = m(t)$ and $n = n(t)$ be the functions of Theorem 1. Then

$$\pi^2 \bar{\sigma}_{\mu\nu}(x,y) = \int_0^{1/m} du \int_{1/m}^{\pi} \{\quad\} dv + \int_0^{1/n} dv \int_{1/m}^{\pi} \{\quad\} du$$
$$+ \int_{1/m}^{\pi} du \int_{1/n}^{\pi} \{\quad\} dv + \int_0^{1/m} du \int_0^{1/n} \{\quad\} dv$$

where the curly brackets $\{\quad\}$ stand for the integrand on the right of (15). Using the inequalities (14) and observing that for each t

$$\lambda^{-1} \leq \mu/m \leq \lambda, \qquad \lambda^{-1} \leq \nu/n \leq \lambda,$$

we immediately find that

$$0 \leq \bar{\sigma}_{\mu\nu}(x,y) \leq A\lambda^2 mn^{-1} \int_0^{1/m} du \int_{1/n}^{\pi} v^{-2} \, | f(x+u, y+v) | \, dv$$
$$+ A\lambda^2 m^{-1} n \int_0^{1/n} dv \int_{1/m}^{\pi} u^{-2} \, | f(x+u, y+v) | \, du$$
$$+ A\lambda^2 m^{-1} n^{-1} \int_{1/m}^{\pi} \int_{1/n}^{\pi} u^{-2} v^{-2} \, | f(x+u, y+v) | \, du dv$$
$$+ A\lambda^2 mn \int_0^{1/m} \int_0^{1/n} | f(x+u, y+v) | \, du dv$$
$$= A\lambda^2 P_t(x,y) + A\lambda^2 Q_t(x,y) + A\lambda^2 R_t(x,y) + A\lambda^2 S_t(x,y),$$

say.

Let

$${}^*P(x,y) = \operatorname*{Sup}_t P_t(x,y) = \operatorname*{Sup}_t mn^{-1} \int_0^{1/m} du \int_{1/n}^{\pi} v^{-2} \, | f(x+u, y+v) | \, dv$$

and let us similarly define ${}^*Q(x,y)$, ${}^*R(x,y)$, ${}^*S(x,y)$.

LEMMA 4. *For $(x,y) \in Q$, each of the functions ${}^*P$, ${}^*Q$, ${}^*R$, ${}^*S$ is majorized by $Af^*(x,y)$, where f^* is the function of Lemma 3 formed with $h(t) = 1/m(t)$, $k(t) = 1/n(t)$.*

Proof. Let us consider the integers $I = I(t)$, $J = J(t)$ defined by the conditions

$$\pi \leq 2^I/m < 2\pi, \qquad \pi \leq 2^J/n < 2\pi.$$

Thus

$$P_t(x,y) = mn^{-1} \sum_{j=1}^{J} \int_0^{1/m} dv \int_{2^{j-1}/n}^{2^j/n} v^{-2} \, | f(x+u, y+v) | \, dv$$
$$\leq 4mn \sum_{j=1}^{J} 2^{-2j} \int_0^{1/m} du \int_{2^{j-1}/n}^{2^j/n} | f(x+u, y+v) | \, dv \leq 16 \sum_{j=1}^{J} 2^{-j} f_{\bullet}^{1,2^j}(x,y),$$

where the function $f_{\bullet}^{\alpha,\beta}(x,y)$ is formed with $h(t) = 1/m(t)$, $k(t) = 1/n(t)$. From the definition of $f^*(x,y)$ it follows immediately that

$$P_t(x,y) \leq 16 f^*(x,y) \sum_{1}^{\infty} 2^{-j/2}, \qquad {}^*P(x,y) \leq Af^*(x,y).$$

It is clear that the last inequality holds for ${}^*Q(x,y)$. As to ${}^*R$, we observe that

$$R_t(x,y) \leq m^{-1}n^{-1}\sum_{i,j=1}^{I,J}\int_{2^{i-1}/m}^{2^i/m}\int_{2^{j-1}/n}^{2^j/n} u^{-2}v^{-2}\,|\,f(x+u,y+v)\,|\,dudv$$

$$\leq 16mn\sum_{i,j=1}^{I,J} 2^{-2(i+j)}\int_{-2^i/m}^{2^i/m}\int_{-2^i/n}^{2^i/n}|\,f(x+u,y+v)\,|\,dudv$$

$$\leq 64\sum_{i,j=1}^{I,J} 2^{-(i+j)} f_*^{2^i,2^j}(x,y)$$

$$\leq 64 f^*(x,y)\sum_{i,j=1}^{\infty} 2^{-(i+j)/2},$$

so that $*R(x,y) \leq Af^*(x,y)$. Since $*S(x,y) \leq 4f_*(x,y) \leq 4f^*(x,y)$, Lemma 4 is proved.

Let us introduce the function

$$(16) \qquad \sigma^*_\lambda(x,y;f) = \operatorname*{Sup}_{\mu,\nu}\,|\,\sigma_{\mu\nu}(x,y;f)\,|\,,$$

where $\mu \geq 1$ and $\nu \geq 1$ satisfy (3). From the formulas (13) and (15) we see that $|\,\sigma_{\mu\nu}\,|$ is majorized by a sum of four integrals typical of which is $\bar\sigma_{\mu\nu}$. Since

$$\bar\sigma_{\mu\nu}(x,y;f) \leq A\lambda^2\{*P(x,y) + *Q(x,y) + *R(x,y) + *S(x,y)\},$$

Lemma 4 and Lemma 3 lead to the following

LEMMA 5. *The function $\sigma^*_\lambda(x,y)$ is majorized by $A\lambda^2 f^*(x,y)$, where f^* is the same as in Lemma 4. The set of points $(x,y)\in Q$ at which $\sigma^*_\lambda(x,y;f) > \xi > 0$ is of measure not exceeding*

$$A\lambda^2\xi^{-1}\int\int_Q |\,f\,|\,dxdy.$$

Theorem 1 is a corollary of Lemma 5. For let us consider a decomposition

$$(17) \qquad f = f_1 + f_2,$$

where $f_1(x,y)$ is a trigonometric polynomial, and the integral $\int\int_Q |\,f_2\,|\,dxdy$ is arbitrarily small. Given an arbitrary $\delta > 0$, we may assume, by virtue of Lemma 2, that the set of points (x,y) at which $\sigma^*_\lambda(x,y;f_2)$ exceeds it is of measure less than δ. Since $\sigma_{\mu\nu}(x,y;f_1)$ tends (uniformly) to $f_1(x,y)$, and since $|\,\sigma_{\mu\nu}(x,y;f_2)\,|$ is less than δ outside a set of measure $< \delta$, it follows without difficulty that $\sigma_{\mu\nu}(x,y;f) = \sigma_{\mu\nu}(x,y;f_1) + \sigma_{\mu\nu}(x,y;f_2)$ tends to $f(x,y)$ almost everywhere, provided conditions (3) are satisfied.

3. For certain applications one needs extensions of Theorems 1 and

to Fourier-Stieltjes series, that is to series (1) whose coefficients are represented by Stieltjes integrals

$$c_{mn} = (2\pi)^{-2} \int\int_Q e^{-i(mx+ny)} dF(E).$$

Here, as before, Q denotes the square $-\pi \leq x \leq \pi$, $-\pi \leq y \leq \pi$, and $F(E)$ is an additive function of sets. The arithmetic means of the series (1) are then represented by the formula

$$\sigma_{\mu\nu}(x, y; dF) = \pi^{-2} \int\int_Q K_\mu(x-u)K_\nu(y-v)dF(E). \tag{19}$$

As is very well known, the function $F(E)$ has almost everywhere a finite derivative $f(x, y)$, and $f(x, y)$ is also the derivative of the absolutely continuous component of $F(E)$.

THEOREM 3. *Let $F(E)$ be an additive function of sets in Q, and let $f(x, y)$ be the derivative of F. Then, at almost every point*

$$\sigma_{\mu\nu}(x, y; dF) \to f(x, y)$$

under the same conditions for μ and ν as in Theorem 1.

In the same sense, Theorem 2 remains valid for Fourier-Stieltjes series.

It is again sufficient to prove the part of Theorem 3 concerning the arithmetic means. A perusal of the proofs of Lemmas 2, 3, 4, 5 shows that these remain valid in the new case. In particular (compare Lemma 5) the set of points $(x, y) \in Q$ at which $\sigma^*_\lambda(x, y; dF) > \xi > 0$ is of measure not exceeding

$$A\lambda^2\xi^{-1} \int\int_Q |dF(E)|. \tag{20}$$

This immediately shows that at almost every point the numbers $\sigma_{\mu\nu}(x, y; dF)$ are bounded. To prove, however, that they tend to $f(x, y)$ we have to adopt a method slightly different from the one used before, since no decomposition corresponding to (17) can be used for singular mass distributions.

Let $F = F_1 + F_2$ be the decomposition of F into its absolutely continuous and singular parts. Thus $\sigma_{\mu\nu}(x, y; dF) = \sigma_{\mu\nu}(x, y; dF_1) + \sigma_{\mu\nu}(x, y; dF_2)$. Since $\sigma_{\mu\nu}(x, y; dF_1) = \sigma_{\mu\nu}(x, y; f) \to f(x, y)$ almost everywhere, it is enough to prove that $\sigma_{\mu\nu}(x, y; dF_2) \to 0$ almost everywhere. We first prove the following lemma.

LEMMA 6. *Suppose that the function F of Theorem 3 has the property*

that $\iint_R |dF(E)| = 0$, *where* R *is a rectangle* $\alpha < x < \alpha'$, $\beta < y < \beta'$. *Then at almost every point of* R *we have* $\sigma_{\mu\nu}(x, y; dF) \to 0$ *as* μ, ν *tend to* $+\infty$ *independently of each other.*

Proof. In the formula (19) we can now integrate over the set $Q - R$. If $(x, y) \epsilon R$, there is an $\eta > 0$ such that for every $(u, v) \epsilon Q - R$ at least one of the inequalities $|x - u| \geq \eta$, $|y - v| \geq \eta$ is satisfied. Since $K_\mu(t)$ tend uniformly to 0 if $\eta \leq |t| \leq \pi$, it follows that

$$|\sigma_{\mu\nu}(x, y; dF)|$$

$$\leq o(1) \iint_{Q-R} K_\mu(x-u) |dF(E)| + o(1) \iint_{Q-R} K_\nu(y-v) |dF(E)|$$

$$\leq o(1) [\iint_Q K_\mu(x-u) |dF(E)| + \iint_Q K_\nu(y-v) |dF(E)|],$$

and it is enough to show that the first of the integrals in square brackets is bounded for almost every x, and the second for almost every y. It suffices to consider the first integral. It can be written

$$\int_{-\pi}^{\pi} K_\mu(x-u) d\chi(u),$$

if $\chi(u)$ denotes the integral $\iint |dF|$ extended over the rectangle $-\pi \leq x \leq u$, $-\pi \leq y \leq \pi$. The last integral is the $(C, 1)$ mean of the Fourier-Stieltjes series of $d\chi(u)$, and so is bounded (indeed, tends to a limit) for almost every x. This completes the proof of Lemma 6.

Let us now revert to the singular function $F_2(E)$. It is well known that, given any number $\epsilon > 0$, we can find an open set $O \subset Q$, of measure differing from that of Q as little as we please and such that $\iint_O |dF_2(E)| < \epsilon$. Let us write $F_2(E) = F_2(OE) + F_2(E - O) = F_3(E) + F_4(E)$, say. By Lemma 6, $\sigma_{\mu\nu}(x, y; dF_4)$ converges to 0 almost everywhere in O. The set of points of Q, and *a fortiori* the set of points of O, at which $\sigma^*_\lambda(x, y; dF_3) > \epsilon^{1/2}$, is of measure

$$\leq A\lambda^2\epsilon^{-1/2} \iint_Q |dF_3| = A\lambda^2\epsilon^{-1/2} \iint_O |dF_2| < A\lambda^2\epsilon^{1/2} .$$

Thus, if we exclude from O a subset of measure $< A\lambda^2\epsilon^{1/2}$, at the remaining points of O the least upper bound of the numbers $|\sigma_{\mu\nu}(x, y; dF_2)|$ is $< \epsilon$. Since both ϵ and $|Q - O|$ can be arbitrarily small, $\sigma_{\mu\nu}(x, y; dF_2)$ tends to 0 almost everywhere in Q. This completes the proof of Theorem 3.

From the extension of Lemma 5 to Fourier-Stieltjes series (cf. 20)), we see that the function $\sigma^*_\lambda(x, y; dF)$ is integrable in any positive power less than 1. More precisely,

THEOREM 4. *Under the assumptions of Theorem 3, and for* $0 < p < 1$,

$$\{\int\int_Q [\sigma^*_\lambda(x, y; dF)]^p dx\}^{1/p} \leq (A/(1-p))\lambda^2 \int\int_Q |dF| \tag{21}$$

$$\int\int_Q [\{\sigma_{\mu\nu}(x, y; dF) - f(x, y)| ^p dx \to 0. \tag{22}$$

The corresponding result holds for Abel means.

So far we have discussed, for simplicity, the case of double Fourier series. The results corresponding to Theorems 1, 2, 3, 4 hold, however, for Fourier series (or Fourier-Stieltjes series)

$$\sum_{n_1,\cdots,n_k=-\infty}^{+\infty} c_{n_1\ldots n_k} e^{i(n_1x_1+\ldots+n_kx_k)} \tag{23}$$

of functions of k variables. So does, as is well known (see Jessen, Marcinkiewicz and Zygmund [2]), Theorem B, which then asserts that if not only the function $f(x_1, \cdots, x_k)$ is integrable, but also $|f|(\log^+|f|)^{k-1}$, then the (C, 1) means

$$\sigma_{n_1\ldots n_k}(x_1, x_2, \cdots, x_k; f)$$
$$= \sum_{\nu_1,\ldots,\nu_k=-n_1,\ldots,-n_k}^{n_1,\ldots,n_k} (1 - \frac{|\nu_1|}{n_1+1}) \cdots (1 - \frac{|\nu_k|}{n_k+1}) e^{i(\nu_1x_1\ldots+\nu_kx_k)}$$

of the series (23) converge almost everywhere to $f(x_1, \cdots, x_k)$ as $n_1, \cdots, n_k$ tend to $+\infty$ independently of one another. The theorem that follows is intermediate between the latter result and the extension of Theorem 1 to the case of k variables.

THEOREM 5. *Let $f(x_1, x_2, \cdots, x_k)$ be a function of $k \geq 2$ variables, of period 2π with respect to each x. Suppose that r is an integer satisfying $0 \leq r \leq k-1$, and that the function $|f|(\log^+|f|)^r$ is integrable. Let $s = k - r$, let $\lambda \geq 1$ be fixed, and let $n_1(t), n_2(t), \cdots, n_s(t)$ be non-negative, non-decreasing, integer-valued functions of t tending to $+\infty$ with t. Then at almost every point the means $\sigma_{\mu_1\ldots\mu_r\nu_1\ldots\nu_s}(x_1, x_2, \cdots, x_k; f)$ tend to $f(x_1, x_2, \cdots, x_k)$, as $\mu_1, \cdots, \mu_r, \nu_1, \cdots, \nu_s$ tend to infinity, provided that $\nu_1, \cdots, \nu_s$ tend to infinity in such a way that*

$$\lambda^{-1}n_j(t) \leq \nu_j \leq \lambda n_j(t) \qquad (j = 1, 2, \cdots, s).$$

The corresponding result for Abel means is also true.

Proof. Given an integrable function $g(x)$ of period 2π, let

$$g_*(x) = \operatorname*{Sup}_{0<|h|\leq\pi} \left| 1/h \int_0^h |f(x+u)|\, du \right|,$$

and let $\sigma_*(x)$ denote the least upper bound of $|\sigma_n(x)|$, σ_n denoting the (C, 1) means of the Fourier series of f. It is well known (Hardy and Littlewood [1]; Zygmund [6], p. 247-248) that $\sigma_*(x) \leq Ag_*(x)$. It is also known that if $|g|(\log^+|g|)^\alpha$ is integrable, so is $g_*(\log^+ g_*)^{\alpha-1}$ and

$$(24) \qquad \int_0^{2\pi} g_*(\log^+ g_*)^{\alpha-1} dx \leq A_\alpha \int_0^{2\pi} |g|(\log^+|g|)^\alpha dx + A_\alpha \qquad (\alpha \geq 1)$$

where A_α denotes a constant depending on α only.

Let $\sigma^*_\lambda(x_1, x_2, \cdots, x_k)$ denote the least upper bound of the numbers $\sigma_{\mu_1\ldots\mu_r\nu_1\ldots\nu_s}(x_1, \cdots, x_k)$ under the conditions on $\mu_1, \mu_2, \cdots, \mu_r, \nu_1, \cdots, \nu_s$ expressed in the statement of Theorem 5. It is enough to prove that for every p, $0 < p < 1$, we have

$$(25) \qquad \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} dx_1 \cdots dx_r \left\{ \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \sigma^{*}{}_\lambda^p dx_{r+1} \cdots dx_k \right\}^{1/p}$$
$$\leq \frac{A_r\lambda^s}{1-p} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |f|(\log^+|f|)^r dx_1 \cdots dx_k + \frac{A_r\lambda^s}{1-p}.$$

That this inequality implies the boundedness almost everywhere of the numbers $\sigma_{\mu_1\ldots\nu_s}(x_1, \cdots, x_k)$ is clear. In order to prove that it also implies $\sigma_{\mu_1\ldots\nu_s}(x_1, \cdots, x_k) \to f(x_1, \cdots, x_k)$ almost everywhere, we proceed in a familiar way. First of all, we fix p and apply (25) to the function Mf, where M is a positive constant so large that in the resulting inequality

$$(26) \qquad \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} dx_1 \cdots dx_r \left\{ \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \sigma^{*}{}_\lambda^p dx_{r+1} \cdots dx_k \right\}^{1/p}$$
$$\leq \frac{A_r\lambda^s}{1-p} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |f|(\log^+|Mf|)^r dx_1 \cdots dx_k + \frac{A_r\lambda^s}{M(1-p)}$$

the last term $A_r\lambda^s/M(1-p)$ on the right is $< \epsilon/2$. Then we make a decomposition $f = f' + f''$, where f' is a trigonometric polynomial in $x_1, \cdots, x_k$, and the first term on the right of (25), with f replaced by f'', is also $< \epsilon$. The final steps may be left to the reader.

Let us revert to (25). Let us fix $x_2, \cdots, x_k$, and let $f_1(x_1, x_2, \cdots, x_k)$

be obtained from $f(x_1, x_2, \cdots, x_k)$ in the same way as $g_*(x_1)$ is obtained from $g(x_1)$. Similarly we obtain $f_2(x_1, \cdots, x_k)$ from $f_1(x_1, \cdots, x_k)$, this time fixing $x_1, x_3, \cdots, x_k$, and so on. Denoting for simplicity integrals extended over the k dimensional cell $Q(|x_j| \leq \pi,\ j = 1, 2, \cdots, k)$, by $\int_{Q_k} \cdots d\omega_k$, we get from (24)

$$(27) \qquad \int_{Q_k} f_r d\omega_k \leq A \int_{Q_k} f_{r-1} \log^+ f_{r-1} d\omega_k + A \leq \cdots$$

$$\leq A_r \int_{Q_k} |f| (\log^+ |f|)^r d\omega_k + A_r,$$

so that f_r is integrable. Let us now observe that

$$|\sigma_{\mu_1 \ldots \nu_s \nu_1 \ldots \nu_s}(x_1, \cdots, x_r, y_1, \cdots, y_s)|$$

$$\leq \pi^{-k} \int_{Q_k} |f(x_1 + u_1, \cdots, y_s + v_s)| \prod_{i=1}^{r} K_{\mu_i}(u_i) \prod_{j=1}^{s} K_{\nu_j}(v_j) d\omega_k$$

$$\leq A \int_{Q_{k-1}} f_1(x_1, x_2 + u_2, \cdots, y_s + v_s) \prod_{i=2}^{r} K_{\mu_i}(u_i) \prod_{j=1}^{s} K_{\nu_j}(v_j) d\omega_{k-1}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\leq A_r \int_{Q_s} f_r(x_1, \cdots, x_r, y_1 + v_1, \cdots, y_s + v_s) \prod_{j=1}^{s} K_{\nu_j}(v_j) d\omega_s.$$

The last integral, multiplied by π^{-k}, represents the arithmetic mean of the Fourier series of the function $f_r(x_1, \cdots, x_r, y_1, \cdots, y_s)$ of the variables $y_1, \cdots, y_s$ (so that $x_1, \cdots, x_r$ are constants). Using the analogue of Theorem 4 (inequality (21)) for the s dimensional case and F absolutely continuous, integrating the result with respect to $x_1, \cdots, x_r$, and applying (27), we get (25). This completes the proof of Theorem 5.

Remarks. a) Inequality (25) implies that under the assumptions of Theorem 5,

$$\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} dx_1 \cdots dx_r \{ \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |\sigma_{\mu_1 \ldots \nu_s}(x_1, \cdots x_k)$$

$$- f(x_1, \cdots, x_k)|^p dx_{r+1} \cdots dx_k \}^{1/p} \to 0$$

and *a fortiori*,

$$\int_{Q_k} |\sigma_{\mu_1 \ldots \nu_s}(x_1, \cdots, x_k) - f(x_1, \cdots, x_k)|^p d\omega_k \to 0.$$

b) The part of Theorem 5 pertaining to Abel summability, is imme-

diately extensible to non-tangential paths, as in Theorem B (see Marcinkiewicz and Zygmund [3]) and Theorem 2. Remark a) applies also to that case.

c) The results hold for fractional summability $(C, \alpha_1, \cdots, \alpha_k)$ if all the α_j are positive.

d) In Theorem 5 we actually have $r + 1$ indices tending to $+\infty$ independently of one another. This is easily seen if one of the functions $n_1(t), \cdots, n_s(t)$ is taken as a new independent variable, instead of t.

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ON RIESZ SUMMABILITY AND SUMMABILITY BY DIRICHLET'S SERIES.*

By C. T. Rajagopal.

Addendum and Corrigendum.

Dr. L. S. Bosanquet has pointed out to me that Corollary 2 (This Journal, vol. 59, pp. 374-5) is incorrectly stated. The corollary is a generalization of only the sufficiency part of Schnee's theorem and should run:

Sufficient conditions for the $R(\lambda_n, k)$-summability $(k \geq 0)$ *of* $\sum_{\nu=1}^{\infty} a_\nu$ *to* s *are* (i) *and* (ii′). *If* $F(t)$ *converges for* $t > 0$, *the conditions* $\lim_{t \to +0} F(t) = s$, $B_k(x) = o(x^{k+1})$ *as* $x \to \infty$, *are necessary for the* $R(\lambda_n, k)$*-summability* $(k > 0)$ *of* $\sum_{\nu=1}^{\infty} a_\nu$.

Dr. Bosanquet has also kindly suggested the following as a generalization of Schnee's theorem.

Corollary 3. *Necessary and sufficient conditions for the* $R(\lambda_n, k)$*-summability* $(k \geq 0)$ *of* $\sum_{\nu=1}^{\infty} a_\nu$ *to* s *are*

(i*) $\dfrac{t^{k+1}}{\Gamma(k+1)} \displaystyle\int_0^{\infty} A_k(u) e^{-ut} du$ *converges (absolutely) for* $t > 0$ *and tends to* s *as* $t \to +\infty$.

(ii*) $$B_k(x) = o(x^{k+1}), \qquad x \to \infty.$$

Further, in the special case in which k *is an integer,* (ii*) *may be replaced by*

(ii**) $$B_k(x) = o(\lambda_n x^k), \qquad \lambda_n \leq x < \lambda_{n+1}.$$

Proof. For any $k \geq 0$, the necessity of (i*) is easily proved and that of (ii) follows from the definition of $B_k(x)$ and the fact that both $A_k(x)/x^k$ and $A_{k+1}(x)/x^{k+1}$ tend to s as $x \to \infty$.

To prove the sufficiency of the conditions (i*) and (ii*), for any $k \geq 0$, we note that (i*) implies the validity of Lemma 4 and that of the proof in **1.2**. Consequently (i*) leads to $A_{k+1}(x)/x^{k+1} \to s$ and thence, in conjunction with (ii*), to $A_k(x)/x^k \to s$.

The replacement of (ii*) by (ii**), when k is integral, is justified by the fact that the latter condition implies the former whether k is integral or not, while the former implies the latter in the special case of integral k as we can see by putting $\mu = k$, $p = 1$ and taking o instead of O in the following theorem of Dr. L. S. Bosanquet which is to be published in the *Journal of the London Mathematical Society*:

If $B_k(x) = O(x^{k+p})$, *where* k *is a positive integer and* $k + p \geq 0$, *then, for* $\mu = 0, 1, \cdots, k$,

$$B_\mu(x) = O\{x^\mu \lambda_n{}^p \left(\frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_n}\right)^{k-\mu}\}, \qquad \lambda_n \leq x < \lambda_{n+1}.$$

It may be observed that the special case of Corollary 3 reduces to Schnee's theorem in the usual form when $k = 0$; also that, in any case, we can obtain a variant of the corollary with the integral in (i*) replaced by $F_k(t) = t \int_0^\infty \frac{A_k(u)}{u^k} e^{-ut} du$.

I take this opportunity to draw the attention of the reader to a misprint in condition (ii) of Corollary 1, p. 374. The restriction on $B_k(x)$ should be

$$B_k(x) \geq -K\lambda_n{}^{k+1}, \text{ not } B_k(x) = -k\lambda_n{}^{k+1}.$$

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ON MÖBIUS' INVERSION.*

By Philip Hartman and Aurel Wintner.

1. Let $x = (x(1), x(2), \cdots)$ and $y = (y(1), y(2), \cdots)$ be two vectors with an infinity of components. Let E and M denote the infinite matrices of the linear substitutions

$$(1) \qquad \mathrm{E}: \quad \sum_{m=1}^{\infty} x(nm) = y(n), \qquad (n = 1, 2, \cdots),$$

and

$$(2) \qquad \mathrm{M}: \quad \sum_{m=1}^{\infty} \mu(m) y(nm) = x(n), \qquad (n = 1, 2, \cdots),$$

respectively, where nm or mn denotes the product of m and n, and $\mu(m)$ is the Möbius factor.

Möbius' formal rule for the inversion of the Eratosthenian sieve process states that the linear substitutions (1), (2) are inverses of each other. Correspondingly, if (1) is written as $\mathrm{E}x = y$ and (2) as $\mathrm{M}y = x$, then $x = \mathrm{M}y$ will be called the Möbius solution of $\mathrm{E}x = y$, where y is given and x is unknown, and a corresponding manner of speaking will be used if E and M, and x and y, are interchanged.

In a paper appearing in vol. 68 (1946), pp. 321-339, of this Journal, the legitimacy of Möbius' formal inversion was considered. It was shown that if a vector $z = (z(1), z(2), \cdots)$ is called *regular* when

$$\sum_{n=1}^{\infty} |z(n)| < \infty,$$

then neither of the Möbius inversions need be legitimate if nothing but the regularity of the respective *solution* vectors, $z = x$ or $z = y$, is assumed; cf. theorems (IV) and (VII), *loc. cit.* On the other hand, various sufficient criteria were developed under which the Möbius inversions are legitimate. The present paper contains further results in the latter direction.

2. The above definition of the "regularity" of a vector is justified by the following fact:

(i) *For a given vector y, the system $\mathrm{E}x = y$ cannot have more than one regular solution $x = x_y$.*

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* Received February 13, 1947.

This is theorem (V), *loc. cit.*, where it is explained that this statement is equivalent to a result of Haar concerning the homogeneous system $\mathrm{E}x = 0$.

It should be emphasized that only the regularity of x is assumed in (i), whereas the given vector y can be arbitrary.

As to the dual of (i), viz., the statement which results if $\mathrm{E}x = y$ is replaced by its formal inverse $\mathrm{M}y = x$, the situation is as follows: *Loc. cit.*, p. 335, a partial dual of (i) was verified, and the question as to its complete dual was left as a desideratum. It will now be shown that the complete dual of (i) is true:

(ii) *For a given vector x, the system $\mathrm{M}y = x$ cannot have more than one regular solution $y = y_x$.*

In (ii), no restriction is placed on the given vector x.

A corollary of (ii) is the following dual of Haar's result: $y = 0$ *is the only regular solution of the homogenous Möbius equations* $\mathrm{M}y = 0$. Needless to say, this corollary of (ii) contains (ii) itself.

The idea of the proof of (ii) is somewhat similar to that of Haar's dual of the last italicized statement (that is, of the fact that $x = 0$ is the only regular solution of the homogeneous Eratosthenian equations $\mathrm{E}x = 0$). The formal details turn out to be more elaborate than in Haar's case.

3. In view of the negative results (IV) and (VII), proved *loc. cit.*, there is a problem concerning the *consistency* of regular solutions and Möbius solutions when both exist. Without any restriction, this question will be left undecided, as it was *loc. cit.* However, some information can be obtained in this regard by the method proving (ii). In fact, a slight modification of the procedure, which would prove (ii) directly, also leads to the following theorem:

(ii*) *For a given regular x, the system $\mathrm{M}y = x$ cannot have a regular solution y distinct from the Möbius vector $\mathrm{E}x$, and the latter represents a solution y of $\mathrm{M}y = x$ whenever the vector $\mathrm{E}x$ is regular.*

In other words, if x is a given regular vector, then, according as the vector $\mathrm{E}x$ is or is not regular, the system $\mathrm{M}y = x$ has the unique regular solution $y = \mathrm{E}x$ or no regular solution at all. The point in this alternative is the fact, mentioned before, that the concepts of Möbius solutions and regular solutions are, in general, distinct. The italicized statement following (ii), which is equivalent to (ii), makes it particularly clear that (ii*) is a refinement of (ii); so that only (ii*) will have to be proved.

There arises the question whether or not (i) can be refined to a theorem,

say (i*), in the same way as (ii) is refined to (ii*). The answer turns out to be affirmative:

(i*) *For a given regular y, the system* $\mathrm{E}x = y$ *cannot have a regular solution x distinct from the Möbius vector* $\mathrm{M}y$, *and the latter represents a solution x of* $\mathrm{E}x = y$ *whenever the vector* $\mathrm{M}y$ *is regular.*

The proof of (i*) will depend on an adaptation of Haar's proof for the uniqueness of the regular solution of the homogeneous system $\mathrm{E}x = 0$.

It is worth mentioning that (i*) and (ii*), respectively, make clear the methodical rôle of the steps by means of which (VIII) and (IX) have been verified *loc. cit.*

4. In order to prove (ii*), suppose that x is a given regular vector with reference to which the system $\mathrm{M}y = x$ has at least one regular solution, y. It will be shown that this y must then be the vector $\mathrm{E}x$. This will imply the first assertion of (ii*).

It will be sufficient to prove that $y(1)$, the first component of the vector y, is the first component of the vector $\mathrm{E}x$; i. e., that (1) must then hold for $n = 1$. In fact, suppose that this has been deduced from the system (2). Then it can be applied to the system which results from the full system (2) when n, in (2), is restricted to multiples of a fixed positive integer, say of i; that is, to the system

$$\sum_{m=1}^{\infty} \mu(m) y(mni) = x(ni), \qquad (n = 1, 2, \cdots).$$

Since the system which corresponds to the latter system in the same way as (1) corresponds to (2) is

$$\sum_{m=1}^{\infty} x(mni) = y(ni), \qquad (n = 1, 2, \cdots), \tag{3}$$

and since the assertion is supposed to be true for $n = 1$, it follows that

$$\sum_{m=1}^{\infty} x(mi) = y(i).$$

Since this means that (1) itself is true for $n = i$ and, since i is arbitrary, it is seen that it is sufficient to prove (1) for $n = 1$.

Let j and k be two positive integers and let the equations (2) be summed over those values of the index n which are composed of the first j primes $p_1 = 2$, $p_2 = 3, \cdots$ with multiplicities not exceeding k (the value $n = 1$ is included). This gives

$$(4) \qquad \sum_{h_1=0}^{k} \cdots \sum_{h_j=0}^{k} x(p_1^{h_1} \cdots p_j^{h_j}) = \sum_{h_1=0}^{k} \cdots \sum_{h_j=0}^{k} \sum_{m=1}^{\infty} \mu(m) y(m p_1^{h_1} \cdots p_j^{h_j}).$$

Since the vector $y = (y(1), y(2), \cdots)$ is supposed to be regular, and since $\mu(m)$ is a bounded function of m, the expression on the right of (4) is absolutely convergent. It can be rearranged into

$$(5) \qquad \sum_{m=1}^{\infty} \sum_{d|m} \mu(m/d) y(m),$$

if the index of the interior summation is restricted to those divisors d of m which satisfy the following restrictions:

$$(6) \qquad d = p_1^{h_1} \cdots p_j^{h_j} \text{ and } 0 \leqq h_i \leqq k, \qquad (i = 1, 2, \cdots, j).$$

If d runs through all divisors of m, then, by the definition of the Möbius function, the sum

$$(7) \qquad \sum_{d|m} \mu(m/d)$$

is 1 or 0 according as $m = 1$ or $m > 1$. On the other hand, $\mu(n)$ is $(-1)^g$ or 0 according as n is the product of g distinct primes or is not square-free. Hence, it is easily verified that, if the summation index is restricted to those divisors d of m which are enumerated under (6), then the corresponding sum (7) is

$$(8) \qquad \mu(m), \qquad (-1)^j \mu(m/P^{k+1}) \text{ or } \qquad 0$$

according as m is relatively prime to the product $P = P_j = p_1 \cdots p_j$, the quotient m/P^{k+1} is an integer relatively prime to P or m is in neither of these cases; that is, according as

$$(9) \qquad (m, P) = 1, \qquad m = lP^{k+1} \textit{ and } (l, P) = 1 \text{ or } \qquad *,$$

where the asterisk denotes the negation of the first two cases of (9) and the three cases of (9) correspond to the respective cases of (8).

It follows that, if d is restricted as in (6), then the double sum (5) is identical with

$$(10) \qquad \sum_{(m,P)=1} \mu(m) y(m) + \sum_{(l,P)=1} \mu(l) y(lP^{k+1}).$$

Since the expression on the left of (4) is identical with the sum (5) restricted by (6), which is the expression (10), and since $|\mu(l)| \leqq 1$, it follows that the absolute value of the difference

$$(11) \qquad \sum_{h_1=0}^{k} \cdots \sum_{h_j=0}^{k} x(p_1^{h_1} \cdots p_j^{h_j}) - \sum_{(m,P)=1} \mu(m) y(m)$$

cannot exceed

$$(12) \qquad \sum_{(l,P)=1} | y(lP^{k+1}) | .$$

Hence, (11) is majorized by

$$\sum_{l=P^{k+1}}^{\infty} | y(l) |, \tag{13}$$

(13) being a majorant of (12).

The indices k and j, hence $P = p_1 \cdots p_j$, had fixed values thus far. Now let $k \to \infty$, while j is fixed. Then (13), hence (11), tends to 0. In view of the regularity of the vector x, this result can be written in the form

$$\sum_{h_1=0}^{\infty} \cdots \sum_{h_j=0}^{\infty} x(p_1^{h_1} \cdots p_j^{h_j}) = \sum_{(m,P)=1} \mu(m) y(m).$$

Since $\mu(1) = 1$ and $|\mu(m)| \leqq 1$, this implies that

$$\left| \sum_{h_1=0}^{\infty} \cdots \sum_{h_j=0}^{\infty} x(p_1^{h_1} \cdots p_j^{h_j}) - y(1) \right| \leqq \sum_{m=p_j+1}^{\infty} | y(m) |. \tag{14}$$

Finally, let $j \to \infty$. Then, since the vector y is supposed to be regular, the expression on the right of (14) tends to 0. On the other hand, since the vector x is supposed to be regular, it is clear that the difference, the absolute value of which occurs on the left of (14), tends to

$$\sum_{m=1}^{\infty} x(m) - y(1),$$

as $j \to \infty$. Hence, the latter expression is 0. This proves that (1) is true for $n = 1$.

5. In order to complete the proof of (ii*), the truth of the following assertion remains to be verified:

(ii bis) *If x and* Ex *are regular vectors, then $y =$* Ex *is a solution y of the system* M$y = x$.

In view of the definitions, (1) and (2), of E and M, the assertion (ii bis) can be formulated as follows: If

$$\sum_{n=1}^{\infty} | x_n | < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \left| \sum_{m=1}^{\infty} x_{mn} \right| < \infty, \tag{15}$$

then

$$\sum_{n=1}^{\infty} \mu(n) \sum_{m=1}^{\infty} x(mni) = x(i) \tag{16}$$

holds for $i = 1, 2, \cdots$. Corresponding to the above reduction of the case on arbitrary n in (1) to the case $n = 1$ of (1), it is sufficient to show that (15) implies

$$\sum_{n=1}^{\infty} \mu(n) \sum_{m=1}^{\infty} x(mn) = x(1), \tag{17}$$

which is the case $i=1$ of (16); in other words, that (17) holds when both x and $\mathbf{E}x$ *are* regular.

Instead of considering, as above, two independent indices, k and j, consider only one of them, j, and put again $P=p_1\cdot\cdot\cdot p_j$. Then, in view of the first of the assumptions (15),

$$(18)\qquad \sum_{h_1=0}^{1}\cdot\cdot\cdot\sum_{h_j=0}^{1}\mu(p_1^{h_1}\cdot\cdot\cdot p_j^{h_j})\sum_{m=1}^{\infty}x(mp_1^{h_1}\cdot\cdot\cdot p_j^{h_j})=\sum_{(m,P)=1}x(m)$$

is a "logical identity" (Sylvester), implied by the definition of the Möbius factor. If $j\to\infty$, then the convergence of the first of the series (15) implies that the sum on the right of (18) tends to $x(1)$. Hence, in order to prove (17), it is sufficient to show that the expression on the left of (18) tends to the expression on the left of (17), as $j\to\infty$. But this becomes clear from the second of the assumptions in (15) if it is observed that, on the one hand, $\mu(n)$ is a bounded function of n and, on the other hand, $\mu(n)$ is 0 when n is not square-free.

This completes the proof of both (ii bis) and (ii*).

6. In order to prove (i*), suppose that y is a given regular vector with reference to which the system $\mathbf{E}x=y$ has at least one regular solution, x. It will be shown that this x must then be the vector $\mathbf{M}y$. This will imply the first assertion of (i*).

The claim is that (2) holds for every n. Corresponding to the above reductions of an arbitrary n to the case of $n=1$, it is sufficient to show that (2) holds for $n=1$.

To this end, choose an arbitrary integer j, put $P=P_j=p_1\cdot\cdot\cdot p_j$, multiply the n-th of the equations (1) by the factor $\mu(n)$ and sum the result over those values of n which are divisors of P. This gives

$$\sum_{h_1=0}^{1}\cdot\cdot\cdot\sum_{h_j=0}^{1}\mu(p_1^{h_1}\cdot\cdot\cdot p_j^{h_j})y(p_1^{h_1}\cdot\cdot\cdot p_j^{h_j})=\sum_{(m,P)=1}x(m).$$

The case $n=1$ of (2) now follows in the same way as (ii bis) did, the last formula line playing the part of (18).

In order to complete the proof of (i*), the truth of the following dual of (ii bis) remains to be verified:

(i bis) *If y and $\mathbf{M}y$ are regular vectors, then $x=\mathbf{M}y$ is a solution x of the system $\mathbf{E}x=y$.*

The proof of this fact can be omitted, since it differs from the proof of the first part of (ii*) only in obvious details.

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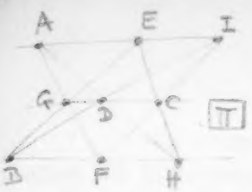

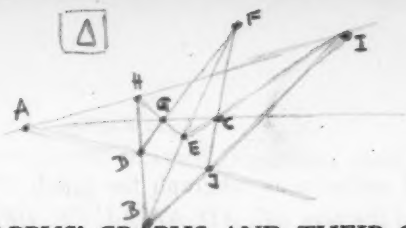

DESARGUES' AND PAPPUS' GRAPHS AND THEIR GROUPS.*

By I. N. Kagno.

1. Introduction.

The graph Δ consisting of the vertices $A, B, C, D, E, F, G, H, I, J$, and the arcs $AC, AD, AG, AH, AI, AJ, BD, BE, BF, BH, BI, BJ, CE, CF, CG, CI, CJ, DF, DG, DH, DJ, EF, EG, EH, EI, FG, FJ, GH, HI, IJ$, is called Desargues' Graph. The graph Π consisting of the vertices $A, B, C, D, E, F, G, H, I$, and the arcs $AD, AE, AF, AG, AH, AI, BD, BE, BF, BG, BH, BI, CD, CE, CF, CG, CH, CI, DG, DH, DI, EG, EH, EI, FG, FH, FI$, is called Pappus' graph. These graphs are defined by Sister Van Straten in an abstract of a paper to be published,[1] and she states the Theorems, (1) Desargues' Graph is an irreducible non-toroidal graph; (2) Pappus' Graph can be imbedded in a torus. The purpose of this note is to present a few additional properties of these graphs, and to derive their groups.

In a previous paper[2] we have defined the group $\mathfrak{G}$ of a graph G as follows; Corresponding to any one-one continuous map of G into itself there is a substitution τ on its vertices $a_1, \cdots, a_n$. Corresponding to the group of all possible maps of G into itself there is a substitution group $\mathfrak{G}$ on the letters $a_1, \cdots, a_n$. $\mathfrak{G}$ is called the group of G, and we say that G has the group $\mathfrak{G}$. In this paper we also defined the adjacency number $I_b{}^a$ of a pair of vertices a, b of G as follows: If G contains the arc ab, then $I_b{}^a = I_a{}^b = 1$. If G does not contain ab, then $I_b{}^a = I_a{}^b = 0$.

2. Desargues' graph.

Theorem 1. *Desargues' Graph Δ cannot be imbedded in a projective plane.*

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* Received November 12, 1946.

[1] Sister Petronia Van Straten, "Toroidal and non-toroidal graphs," *Bulletin of the American Mathematical Society*, vol. 52 (1946), p. 831, Abstract No. 345. She uses a numeral notation to designate the vertices instead of the notation we find it convenient to use here. The full paper is to appear in *Reports of a Mathematical Colloquium* (Notre Dame), No. 8.

[2] "Linear graphs of degree ≤ 6 and their groups," *American Journal of Mathematics*, vol. 68 (1946), pp. 505-520. We shall refer to this paper as [L]. In this paper the author gave an example of a graph of Degree 7, with no vertex of degree < 3, having no non-identical substitution group. It is of interest to note that if the restriction on the degree of the vertices is removed it is possible to find a graph of *fewer* than seven vertices which has no non-identical substitution group. Namely, the graph consisting of the vertices a, b, c, d, e, f, and the arcs $ab, bc, bd, ce, de, cf, ef$, has this property. (We omit the proof, which can readily be supplied by the reader of [L].)

Proof. Δ contains as subgraph the graph Δ^* consisting of the vertices $A, \cdots, J$, and the arcs $AC, AD, AI, AJ, CF, DF, CI, CJ, DH, HI, BD, BJ, IJ, BE, EH, EF$. Δ^* is homeomorphic with the irreducible non-projective-planar graph $G_{iv\text{-}a\text{-}2}$ discussed by the author in a previous paper.[3] Hence Δ is non-projective-planar.

Definition. Let G be a connected graph of n vertices, having no simple loops. Let N be the complete n-point formed by joining each pair of non-adjacent vertices of G by an arc, and let $G' = N$ — the arcs of G. G' will be called the *complement* of G.

THEOREM 2. *The complement of Desargues' Graph is Petersen's Graph.*

Proof. The complement of Δ is the graph Δ' consisting of the vertices of Δ and the arcs $AB, AE, AF, BC, BG, CD, CH, DE, DI, EJ, FH, FI, GI, GJ, HJ$. Let us replace the letters $A, \cdots, J$ by unordered number couples as follows; $A = (1,2)$, $B = (4,5)$, $C = (2,3)$, $D = (1,5)$, $E = (3,4)$, $F = (3,5)$, $G = (1,3)$, $H = (1,4)$, $I = (2,4)$, $J = (2,5)$. Then it will be seen that Δ' is precisely Petersen's Graph.[4]

LEMMA. *If G' is the complement of G, then G and G' have the same group.*

Proof. Let p, q be any pair of vertices. If $I_q^p = 1$, $(= 0)$, in G, then in G', $I_q^p = 0$, $(= 1)$. Now suppose τ is any map of G into itself. Since τ is arc preserving, $I_q^p = I_{\tau(q)}^{\tau(p)}$. Now since in G', I_q^p and $I_{\tau(q)}^{\tau(p)}$ each have the opposite value they have in G, $I_q^p = I_{\tau(q)}^{\tau(p)}$ in G' also. Consequently τ maps G' into itself.[5] Conversely any map of G' into itself also maps G into itself. Hence G and G' have the same groups.

THEOREM 3. *Desargues' Graph has a group of degree ten and order* 120, *which is simply isomorphic with the symmetric group $\mathfrak{S}_5$ on five letters.*

Proof. By the lemma, Δ has the same group as its complement, Petersen's Graph. But Petersen's Graph has the group given in the Theorem.[6]

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[3] "The mapping of graphs on surfaces," *Journal of Mathematics and Physics*, vol. 16 (1937), pp. 46-75; page 66 and plate I on page 62. Note that on page 66 the symbol for $G_{iv\text{-}a\text{-}2}$ is misprinted. It should read $G_{iv\text{-}a\text{-}2} = (B/0/p/pp_{bd}, pp_{dc}, pp_{de})$.

[4] R. Frucht, "Die Gruppe des Petersenschen Graphen . . .," *Commentarii Mathematici Helvetici*, vol. 9 (1937), pp. 217-223.

[5] [L], Theorem 1.

[6] Frucht, *loc. cit.* The group of Δ can be obtained from $\mathfrak{S}_5$ as follows; Replace the letters $A, \cdots, J$ by the number couples as was done in the proof of Theorem 2. Every substitution of $\mathfrak{S}_5$ on the numbers $1, \cdots, 5$ will determine a substitution on the distinct number couples, that is, a substitution on the letters $A, \cdots, J$. $\mathfrak{S}_5$ can be gene-

3. Pappus' graph.

THEOREM 4. *Pappus' Graph* Π *cannot be imbedded in a projective plane.*

Proof. Π contains as subgraph the graph Π^* consisting of the vertices of Π and the arcs AD, AE, AF, BD, BE, BF, CD, CE, CF, GD, GE, GF, HD, HE, HF. Π^* is homeomorphic with the irreducible non-projective-planar graph G_{444-a}.[7] Hence Π is non-projective-planar.

THEOREM 5. *Pappus' Graph has the group*

$\mathfrak{G} = \mathfrak{G}_1 \times \mathfrak{G}_2 \times \mathfrak{G}_3 \times \mathfrak{G}_4 = (ABC)\mathrm{all}(DEF)\mathrm{all}(GHI)\mathrm{all}(ADG \cdot BEH \cdot CFI)$ *of order* 1296.[8]

Proof. We note the following adjacency numbers for Π

$$I_B{}^A = I_C{}^B = I_C{}^A = 0, \qquad I_E{}^D = I_F{}^E = I_F{}^D = 0, \qquad I_H{}^G = I_I{}^H = I_I{}^G = 0,$$

and $I_y{}^x = 1$ for any other pair of vertices of Π.

A set of generators for $\mathfrak{G}$ is $[(ABC), (AB), (DEF), (DE), (GHI), (GH), (ADG)(BEH)(CFI), (AD)(BE)(CF)]$. It can readily be seen that each of these substitutions maps Π into itself. Hence Π is mapped into itself by every substitution of $\mathfrak{G}$.

Suppose Π is mapped into itself by a substitution

$\tau = \begin{pmatrix} A & B & C & D & E & F & G & H & I \\ a & b & c & d & e & f & g & h & i \end{pmatrix} \neq 1.$ We shall show that $\tau \in \mathfrak{G}$.

Case 1. $\tau(A)$ is A, B, or C. Since $I_A{}^B = 0$, $I_A{}^C = 0$, $I_a{}^p = 1$, (where $a = A, B$, or C; $p \neq A, B$, or C), τ cannot carry B or C into D, E, F, G, H, or I. That is, a, b, c is a permutation of A, B, C. The subgroup $\mathfrak{G}_1$ of $\mathfrak{G}$ contains a substitution σ such that $\sigma^{-1} = \begin{pmatrix} a & b & c \\ A & B & C \end{pmatrix}$. Let

$$\alpha = \sigma^{-1}\tau = \begin{pmatrix} A & B & C & D & E & F & G & H & I \\ A & B & C & d & e & f & g & h & i \end{pmatrix}.$$

(a) If $d \neq G$, H, or I, since $I_D{}^m = 0$, $I_d{}^q = 1$, (where $m = E$ or F; $q = G$, H, or I; $d = D$, E, or F), α cannot carry E or F into G, H, or I. That is, d, e, f is a permutation of D, E, F. The subgroup $\mathfrak{G}_2$ of $\mathfrak{G}$ contains a substitution ρ such that $\rho^{-1} = \begin{pmatrix} d & e & f \\ D & E & F \end{pmatrix}$, and the subgroup $\mathfrak{G}_3$ contains a

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rated by the substitutions (1234) and (15). These correspond to the substitutions $(AHEC)(BFJD)(GI)$ and $(AJ)(BH)(FG)$ respectively, which generate the group of Δ.

[7] "The mapping of graphs . . .," *loc. cit.*, page 65 and plate I on page 62.

[8] Where $\mathfrak{G}_4 \equiv \{1, (ADG)(BEH)(CFI), (AGD)(BHE)(CIF), (AD)(BF)(CF), (AG)(BH)(CI), (DG)(EH)(FI)\}$.

substitution μ such that $\mu^{-1}=\begin{pmatrix} g\ h\ i \\ G\,H\,I \end{pmatrix}$. Hence $\mu^{-1}\rho^{-1}\sigma^{-1}\tau=1$ and $\tau=\sigma\rho\mu\,\varepsilon\,\mathfrak{G}$.

(b) If $d=G$, H, or I, since $I_D{}^n=0$, $I_d{}^r=1$, (where $n=E$ or F; $r=D, E$, or F; $d=G, H$, or I), α cannot carry E or F into D, E, or F. That is, d, e, f is a permutation of G, H, I, and consequently g, h, i is a permutation of D, E, F. The subgroup $\mathfrak{G}_3$ of $\mathfrak{G}$ contains a substitution λ_1 such that $\lambda_1{}^{-1}=\begin{pmatrix} d\ e\ f \\ G\,H\,I \end{pmatrix}$, and the subgroup $\mathfrak{G}_2$ contains a substitution λ_2 such that $\lambda_2{}^{-1}=\begin{pmatrix} g\ h\ i \\ D\,E\,F \end{pmatrix}$. Hence $\lambda_1{}^{-1}\lambda_2{}^{-1}\alpha=\begin{pmatrix} A\,B\,C\ D\,E\,F\ G\,H\,I \\ A\,B\,C\ G\,H\,I\ D\,E\,F \end{pmatrix}$. The subgroup $\mathfrak{G}_4$ contains the substitution $\lambda_3=(DG)(EH)(FI)$. Hence $\lambda_3\lambda_1{}^{-1}\lambda_2{}^{-1}\alpha=\lambda_3\lambda_1{}^{-1}\lambda_2{}^{-1}\sigma^{-1}\tau=1$, and $\tau=\sigma\lambda_2\lambda_1\lambda_3\,\varepsilon\,\mathfrak{G}$.

Case 2. $\tau(A)$ is D, E, or F. Since $I_A{}^k=0$, $I_a{}^s=1$, (where $k=B$ or C; $a=D$, E or F; $s=A$, B, C, G, H, or I), τ cannot carry B or C into A, B, C, G, H or I. That is, a, b, c is a permutation of D, E, F. The subgroup $\mathfrak{G}_2$ of $\mathfrak{G}$ contains a substitution ν such that $\nu^{-1}=\begin{pmatrix} a\ b\ c \\ D\,E\,F \end{pmatrix}$. Let $\beta=\nu^{-1}\tau=\begin{pmatrix} A\,B\,C\ D\,E\,F\ G\,H\,I \\ D\,E\,F\ d\ e\ f\ g\ h\ i \end{pmatrix}$.

(a) If $d=A$, B, or C, since $I_D{}^l=0$, $I_d{}^t=1$, (where $l=E$ or F; $t=D, E, F, G, H$, or I; $d=A$, B, or C), β cannot carry E or F into D, E, F, G, H, or I. That is d, e, f is a permutation of A, B, C. Now $\mathfrak{G}_1$ contains a substitution ϕ_1 such that $\phi_1{}^{-1}=\begin{pmatrix} d\ e\ f \\ A\,B\,C \end{pmatrix}$, and $\mathfrak{G}_3$ contains a substitution ϕ_2 such that $\phi_2{}^{-1}=\begin{pmatrix} g\ h\ i \\ G\,H\,I \end{pmatrix}$, and hence $\phi_2{}^{-1}\phi_1{}^{-1}\beta=\begin{pmatrix} A\,B\,C\ D\,E\,F \\ D\,E\,F\ A\,B\,C \end{pmatrix}$. $\mathfrak{G}_4$ contains a substitution $\phi_3=(AD)(BE)(CF)$. Hence

$\phi_3\phi_2{}^{-1}\phi_1{}^{-1}\beta=\phi_3\phi_2{}^{-1}\phi_1{}^{-1}\nu^{-1}\tau=1$, and $\tau=\nu\phi_1\phi_2\phi_3\,\varepsilon\,\mathfrak{G}$.

(b) If $d=G$, H, or I, since $I_D{}^v=0$, $I_d{}^u=1$ (where $v=E$ or F; $u=A, B, C, D, E$, or F; $d=G$, H, or I), β cannot carry E or F into A, B, C, D, E, or F. That is, d, e, f is a permutation of G, H, I, and consequently g, h, i is a permutation of A, B, C. Now $\mathfrak{G}_3$ contains a substitution θ_1 such that $\theta_1{}^{-1}=\begin{pmatrix} d\ e\ f \\ G\,H\,I \end{pmatrix}$, and $\mathfrak{G}_1$ contains a substitution θ_2 such that $\theta_2{}^{-1}=\begin{pmatrix} g\ h\ i \\ A\,B\,C \end{pmatrix}$ and hence $\theta_2{}^{-1}\theta_1{}^{-1}\beta=\begin{pmatrix} A\,B\,C\ D\,E\,F\ G\,H\,I \\ D\,E\,F\ G\,H\,I\ A\,B\,C \end{pmatrix}$. $\mathfrak{G}_4$ contains a substitution θ_3 such that $\theta_3{}^{-1}=(ADG)(BEH)(CFI)$. Hence $\theta_3{}^{-1}\theta_2{}^{-1}\theta_1{}^{-1}\beta=\theta_3{}^{-1}\theta_2{}^{-1}\theta_1{}^{-1}\nu^{-1}\tau=1$ and $\tau=\nu\theta_1\theta_2\theta_3\,\varepsilon\,\mathfrak{G}$.

Case 3. $\tau(A)$ is G, H, or I. The proof is similar to that of Case 2.

NEW YORK CITY.

THE ISOPERIMETRIC PROBLEM IN THE MINKOWSKI PLANE.*

By Herbert Busemann.

The slow progress in the theory of Finsler spaces as compared to Riemann spaces is partly due to lack of information regarding the corresponding local, that is the Minkowskian, geometry. Those Minkowskian features will contribute most to an understanding of Finsler spaces which are not merely *verbal* generalizations of *known* euclidean statements.[1] The purpose of the present note was originally only to show that the isoperimetric problem (for any dimension) in Minkowski spaces leads to such a feature.

It turned out, however, that the plane problem can be solved in a general form—no longer significant for Finsler spaces—and then exhibits a phenomenon which is of interest for the theory of isoperimetric problems in the calculus of variations. The result seems to indicate that the standard methods may have followed too closely the pattern of the fixed endpoint problem. For that reason the plane case is here presented separately.

The following are the results: let $F(x,y)$ be continuous, positive for $x,y \neq 0$, and positive homogeneous of order 1. The problem, to find among all simple closed curves $x(t), y(t)$ with a given orientation and a given Minkowski length $L = \int F(\dot{x}, \dot{y})\,dt$ one which bounds the greatest (euclidean) area, *has a unique solution* (up to translations) *no matter whether the indicatrix* $C : F(x,y) = 1$ *is convex or not.* For non-convex C the solution is the same as for the boundary $\bar{C}$ of the convex closure of C as indicatrix and *is homothetic to the polar reciprocal* (figuratrix) *of* $\bar{C}$ *with respect to the unit circle rotated through* $\pm \pi/2$.

For Finsler spaces only the case is of interest where C is convex and has the origin as center. Since Finsler or Minkowski area differs from the euclidean area by a constant factor, *the solution of the Minkowskian isoperimetric problem is the same as for the above problem.* In intrinsic Minkowskian terms it may be described as the curve of length L for which as new

* Received January 22, 1947.

[1] Since the euclidean geometry is a special Minkowskian geometry, every theorem on general Minkowskian spaces is a generalization of some euclidean fact.—It has been stated that the so-called relative differential geometry is the differential geometry of Minkowski space. This is however not so. Relative length or area are not Minkowski length or area.

unit circle (indicatrix) perpendicularity (transversality) is the reverse of perpendicularity with respect to C. *It is in general not a Minkowski circle.*[2]

The result for non-convex C suggests the question: what is the relation between the lengths defined by C and $\bar{C}$ as indicatrices respectively. An answer is given in the final section for general n-dimensional Finsler spaces to the effect that *the corresponding Lebesgue lengths are equal.*

1. The form and the uniqueness of the solution for a *convex indicatrix* C can be derived in a few lines from the *Brunn-Minkowski Theory.* If $x = r\cos\theta$, $y = r\sin\theta$ and $F(\cos\theta, \sin\theta) = \rho(\theta)$, then $r = \rho^{-1}(\theta)$ is the equation of C in polar coordinates.

Let D be an analytic closed convex curve which contains the origin in its interior. If $h(x,y)$ is the supporting function of D then $h(\theta) = h(\cos\theta, \sin\theta)$ is the distance of the origin from the tangent of D at the point q where the exterior normal of D has direction θ. The radius of curvature of D at q is $h(\theta) + h''(\theta)$ (see [1, p. 65]) so that the euclidean line element of D at q equals $(h(\theta) + h''(\theta))d\theta$. Therefore the Minkowski length $L(D)$ of D is

$$L(D) = \int_0^{2\pi} (h(\theta) + h''(\theta))\rho(\theta + \delta\pi/2)\,d\theta \tag{1}$$

where $\delta = 1$ ($\delta = -1$) if the orientation of D is positive (negative). From now on we fix the orientation of D and give δ the corresponding value.

If C is convex then $F(x,y)$ is convex, hence $H(x,y) = r\rho(\theta + \delta\pi/2)$ is convex and therefore the supporting function of a certain convex curve K. Then $L(D)$ is by (1) twice the mixed area $A(D,K)$ of D and K (see [4, p. 245]),

$$L(D) = 2A(D,K). \tag{2}$$

Both $L(D)$ and $A(D,K)$ do not change when D or K undergo translations ([1, p. 40]) and depend continuously on D. Since every convex curve can be approximated by analytic convex curves ([1, section 27]), the relation (2) holds for any convex curve D. The Theorem of Brunn-Minkowski yields ([1, p. 97]) that

$$A^2(D,K) \geq A(D)A(K), \tag{3}$$

[2] This fact was first noticed by S. Ulam who found, and communicated to the author, that the solution for $F(x,y) = \max(|x|, |y|)$ is $|x| + |y| = \text{const}$. In this connection it is of interest that *the Minkowski circle is the solution of the problem to find among all curves with a given Minkowski diameter one which bounds the greatest area*, compare [2, section 2].

where $A(D)$ and $A(K)$ are the areas bounded by D and K respectively, and the equality sign holds only when D and K are homothetic. The relations (2) and (3) yield the isoperimetric inequality

$$L^2(D)/A(D) \geq 4A(K). \tag{4}$$

If D is a general simple closed curve with the given orientation, then $L(D)$ is to be defined by the Weierstrass sum. If D is not rectifiable in the euclidean sense, then $L(D) = \infty$. If D is rectifiable and suitably parametrized (for instance by the arc length) then the Weierstrass sum coincides with the Lebesgue integral $\int F(\dot{x}, \dot{y})dt$ along D (see [5, p. 51]). If C is convex the euclidean straight lines are shortest connections. Hence, if D is not convex the boundary of the convex closure of D has by the usual arguments at most the length of D and bounds a greater area, so that (4) holds for any simple closed curve with the given orientation, and the equality still holds only for D homothetic to K.

It is well known that the curve $\bar{K}$ with the supporting function $F(x, y)$ (the so called figuratrix of F) originates from C by a polar reciprocity with respect to the euclidean unit circle (see [4, pp. 146, 147]). Then K is obtained from $\bar{K}$ by a rotation through $-\delta\pi/2$ about O.

2. This way of generating K leads to the property of K which permits the discussion of *non-convex indicatrices.* Call tangent of K a supporting line of K which is at the same time a right hand or left hand tangent of K at one of its common points with K. Then (if C is convex)

(5) No tangent of K is parallel to a radius of C from O to an interior point of a (euclidean) segment ab which lies on C.

For the polars of the different points of ab with respect to $x^2 + y^2 = 1$ pass through one point p and are supporting lines of $\bar{K}$ at p. Hence a supporting line $\bar{S}_x$ of $\bar{K}$ which corresponds to an interior point x of ab cannot be a tangent of $\bar{K}$. After rotation of $\bar{K}$ through $-\delta\pi/2$ this fact becomes (5).

Let now C be not convex and call $\bar{F}(x, y)$ the positive, positive homogeneous function of degree 1 for which $\bar{F}(x, y) = 1$ is the boundary $\bar{C}$ of the convex closure of C. Then $\bar{F}(x, y) \leq F(x, y)$ and if $\bar{L}(D)$ denotes the length of D measured by $\bar{F}$,

$$\bar{L}(D) \leq L(D). \tag{6}$$

If K^* denotes the curve originating from $\bar{C}$ by a polar reciprocity in $x^2 + y^2 = 1$ and rotation through $-\delta\pi/2$, then by (4) and (6)

$$L^2(D)/A(D) \geq \bar{L}^2(D)/A(D) \geq 4A(K)$$

and the equality signs can hold only if D is homothetic to K and $L(D) = \bar{L}(D)$. *The last relation holds always for D homothetic to K.*

For if $u(t), v(t)$ represents K with the euclidean arc length t as parameter, then $\dot{u}$ and $\dot{v}$ exist except for an at most countable number of t's and

$$L(D) = \int F(\dot{u}, \dot{v})dt, \qquad \bar{L}(D) = \int \bar{F}(\dot{u}, \dot{v})dt.$$

If $F(x,y) > \bar{F}(x,y)$, then the ray from O to (x,y) intersects C in a point where C has no supporting line, and therefore $\bar{C}$ in an interior point of a segment ab which is part of $\bar{C}$. The statement (5) shows now that $F(\dot{u}, \dot{v}) = \bar{F}(\dot{u}, \dot{v})$ for any t for which $\dot{u}$ and $\dot{v}$ exist, which proves $L(D) = \bar{L}(D)$. The following has been proved

THEOREM 1. *Let $F(x,y)$ be continuous, positive for $x, y \neq 0$ and positive homogeneous of degree 1. If $r = \bar{\rho}^{-1}(\theta)$ is the polar equation of the boundary $\bar{C}$ of the convex closure of $F(x,y) = 1$, then for any simple closed curve D*

$$L^2(D)/A(D) \geq 2 \int_0^{2\pi} (\bar{\rho}^2(\theta) - \bar{\rho}'^2(\theta))d\theta \tag{7}$$

where $L(D)$ is length in terms of F and $A(D)$ is the euclidean area.

The equality sign holds for positively (negatively) traversed D only if D is homthetic to the polar reciprocal K of C with respect to $x^2 + y^2 = 1$ rotated about O through $-\pi/2(\pi/2)$.

The right side of (7) equals $4A(K)$ by a known integral representation of area in terms of the supporting function, see [1, p. 66]. It is true that this reference states the formula only for analytic curves, but its validity for general convex curves follows from the fact that if the analytic convex curves $r = \rho_\nu^{-1}(\theta)$ tend to $r = \rho(\theta)$, then their derivatives ρ'_ν tend automatically to ρ', see [1, p. 35].

3. Area is defined only for symmetric Finsler spaces [see 2, section 2]. Therefore, as far as Minkowski geometry is concerned, Theorem 1 is interesting only when C is convex and has O as center $(F(x,y) = F(-x, -y))$. In that case the (non-oriented euclidean straight) line g is said to be perpendicular to the line h (or h transversal to g) for the Minkowski geometry with unit circle C, in a formula: $g \perp_C h$, if the parallel to h through O intersects C in those points where C has supporting lines parallel to h. This terminology is due to the fact that the intersection of g and h minimizes the Minkowski

distance of a given point on g from a variable point on h. If K is defined as before, then the statements

(8) $\qquad g \perp_C h$ *and* $h \perp_K g$ *are equivalent*

This may be seen as follows: If p is a point of C and h is a supporting line of C at p let the euclidean normal to h from the origin O intersect h at f and $\bar{K}$ at $\bar{p}$. By the definition of $\bar{K}$ the line Op must intersect a supporting line $\bar{h}$ of $\bar{K}$ at $\bar{p}$ at a right angle. Denote the intersection by $\bar{f}$. If $\bar{p}, \bar{f}, \bar{h}$, are transformed into p_1, f_1, h_1 after rotation through $-\pi/2$ about O, then Op is parallel to h_1 and Op_1 is parallel to h. This means: if $Op = g$ is an arbitrary radius of C and h is a supporting line of C at p then the supporting line h_1 of K parallel to g contains a point p_1 for which $g_1 = Op_1$ is parallel to h, which is equivalent to (8).

Minkowski area (measure) is the euclidean area (Lebesgüe measure) multiplied by π and divided by the euclidean area $A(C)$. Hence, if $M(D)$ denotes the Minkowski area bounded by D, then

$$M(D) = 2\pi A(D) / \int_0^{2\pi} \rho^{-2}(\theta)\, d\theta,$$

where $r = \rho^{-1}(\theta)$ is the polar equation of C. The relation (7) yields then the *Minkowskian Isoperimetric Inequality*

$$(9) \qquad L^2(D)/M(D) \geq \pi^{-1} \int_0^{2\pi} (\rho^2 - \rho'^2)\, d\theta \int_0^{2\pi} \rho^{-2} d\theta,$$

and the equality sign holds only for the curves K which satisfy (8).

The right side of (9) is a Minkowski invariant, hence it does not change if the x, y undergo a non-degenerate affine transformation.

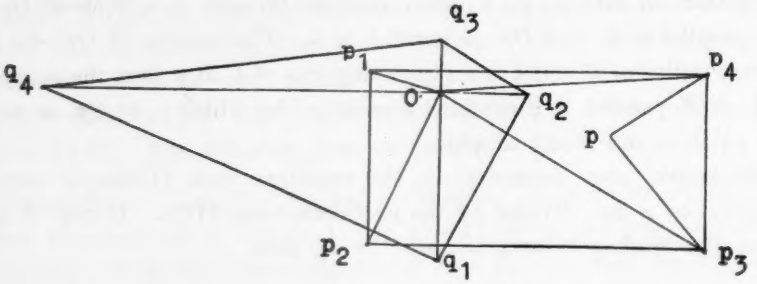
For the theory of Finsler spaces it is interesting to find out in which cases the Minkowski circles sōlve the isoperimetric problem. By (8) the symmetry of the relation $g \perp_C h$ is necessary and sufficient. The analytic equivalent to this statement is that the supporting function $h(\theta)$ of C must be proportional to $\rho(\theta + \pi/2)$. The factor of proportionality is easily evaluated by expressing $A(C)$ first in terms of the radius vector and then in terms of the supporting function of C. It is then found that

$$(10) \qquad h(\theta) = \left[\int_0^{2\pi} \rho^{-2}(\theta)\, d\theta\right]^{1/2} \cdot \left[\int_0^{2\pi} (\rho^2(\theta) - \rho'^2(\theta))\, d\theta\right]^{-1/2} \rho(\theta + \pi/2).$$

All curves with a continuous non-vanishing curvature (the Legendre condition of the calculus of variations) which satisfy (10) have been

determined by Radon in [6]. But there are other curves with this property, for instance the regular n-gons with $n \equiv 2 \bmod 4$.[3]

To have examples for the preceding theory it is useful to construct K for any convex polygon C, not necessarily with O as center. Since the convex closure of any polygon is again bounded by a polygon, this construction will yield K for arbitrary polygonal C. To obtain the counterclockwise solution corresponding to C let $p_1, p_2, \cdots, p_n$ be the vertices in counterclockwise order (see figure). Orient the radii



of C from p_i to O. Then K is the polygon $q_1, \cdots, q_n$ such that the oriented side $q_{i-1}q_i$ is parallel to p_iO and Oq_i is parallel to p_ip_{i+1} (subscripts are to be reduced mod n if necessary). In the figure the polygon K is also the solution for the polygon $C' = (p_1p_2p_3pp_4)$ which has C as boundary of its convex closure. If ab denotes also the euclidean distance of the points a, b and β_i the angle $p_{i-1}Op_i$, the following formula determines the lengths of the sides of K:

$$(11) \qquad \frac{L(K)}{2A(K)} = \text{ᵻƃɪ-ᵻƃ}\, \frac{Op_{i-1}{}^{-1} \sin \beta_{i+1} - Op_i{}^{-1} \sin(\beta_i + \beta_{i+1}) + Op_{i+1}{}^{-1} \sin \beta_i}{\sin \beta_i \sin \beta_{i+1}}.$$

This relation can be obtained through solving the isoperimetric problem for the polygon $p_1, \cdots, p_n$ directly by means of elementary calculus and trigonometry. (11) contains the inequality (3), including the discussion of the equality sign, and yields thus a new elementary proof for the Brunn-Minkowski inequality for polygons. By a limit process the inequality for general convex curves is obtained, but the "only if" part in the condition for the equality sign is lost. For a regular n-gon (11) yields

[3] While this paper was in print a geometric construction for *all* curves C with symmetric $\perp c$ was given by M. M. Day in: "Some characterizations of inner-product spaces," *Transactions of the American Mathematical Society*, vol. 62 (1947), pp. 320-337. The author seems to be unaware of the earlier work of Radon, Helly, and Blaschke on symmetric transversality.

$$L^2(D)/M(D) \geq 4\pi^{-1}n^2 \sin^2(\pi/n)$$

which becomes for $n \to \infty$ the ordinary isoperimetric inequality.

4. The discussion in Section **2** leads to the question: what is for arbitrary D the relation between $L(D)$ and $\bar{L}(D)$. This question will be answered completely in the present section for general n-dimensional Finsler spaces because it throws some light on the implications of the Lebesgue type of definition of length and area.

Let $F(x_1, \cdots, x_n; \alpha^1, \cdots, \alpha^n) = F(x, \alpha)$ be defined and continuous for all contravariant vectors (x, α) of an n-dimensional manifold of class 1, positive for $\alpha \neq 0$, and positive homogeneous of order 1 in the α^i. The Lebesgue F-length of a continuous curve $x(t)$ is defined as

$$\lambda^L_F(x(t)) = \inf [\liminf \int F(x_\nu(t), \dot{x}_\nu(t))dt]$$

for all sequences $\{x_\nu(t)\}$ of curves of class D' which tend to $x(t)$ in the sense of Fréchet.

With $F(x, \alpha)$ we associate a quasiregular integrand $\bar{F}(x, \alpha)$ as follows.[4] For fixed x (in a definite coordinate neighborhood) let $\bar{C}_x$ be the boundary of the convex closure of the indicatrix $F(x, \alpha) = 1$ in α-space. Then $\bar{F}(x, \alpha)$ is the integrand whose indicatrix (in the same x-system) is $\bar{C}_x$. The relation between $L(D)$ and $\bar{L}(D)$ is then determined by the following general theorem.

THEOREM 2. *For any continuous curve* $x(t)$

$$\lambda^L_F(x(t)) = \lambda^L_{\bar{F}}(x(t)).$$

Since $\bar{F}$ is quasiregular the ordinary $\bar{F}$-length $\lambda^L_{\bar{F}}(x(t))$ of a curve[5] coincides with the Lebesgue length $\lambda^L_{\bar{F}}(x(t))$, so that Theorem 2 may also be formulated as

THEOREM 2'. *For any continuous curve* $x(t)$

$$\lambda^L_F(x(t)) = \lambda_{\bar{F}}(x(t)).$$

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[4] For further details see [3]. The following considerations follow closely Section 3 of that paper. The present Theorem 2 is a generalization of Theorem 1, [3, p. 184].

[5] This means again the limit of the Weierstrass sum $\sum \bar{F}(x(t_i), x(t_{i+1}) - x(t_i))$ which for absolutely continuous $x(t)$ coincides with the Lebesgue integral $\int \bar{F}(x, \dot{x})dt$. The F-length λ_F is defined correspondingly. Regarding these questions compare [5, p. 51].

Proof. Since $\bar{F}(x, \alpha) \leq F(x, \alpha)$ and $\lambda_{\bar{F}}$ is lower semicontinuous it follows that for a suitable sequence of curves $x_\nu(t)$ of class D' which tend to x

$$\lambda^{L}_{F}(x) = \lim \lambda_{F}(x_\nu) \geq \liminf \lambda_{\bar{F}}(x_\nu) \geq \lambda_{\bar{F}}(x).$$

Since λ^{L}_{F} is lower semicontinuous it suffices to prove $\lambda_{\bar{F}}(x) \geq \lambda^{L}_{F}(x)$ for x which are of class D'. For then with a suitable sequence of curves of class D' which tend to x

$$\lambda_{\bar{F}}(x) = \lim \lambda_{\bar{F}}(x_\nu) \geq \liminf \lambda^{L}_{F}(x_\nu) \geq \lambda^{L}_{F}(x).$$

Since both lengths are additive it is also sufficient to consider a curve $x(t)$, $a \leq t \leq b$, of class C' which lies entirely in one coordinate neighborhood. If the norm of the partition $[t] = [t_0 = a < t_1 < t_2 < \cdots < t_{N+1} = b]$ tends to 0

$$\Sigma \bar{F}(x(t_i), x(t_{i+1}) - x(t_i)) \rightarrow \lambda_{\bar{F}}(x).$$

The direction of α is called quasiregular for $F(x, \alpha)$ if $F(x, \alpha)$ possesses at its point of the form $\delta\alpha$, $\delta > 0$, a supporting plane. *The relation* $F(x, \alpha) = \bar{F}(x, \alpha)$ *holds if and only if* α *is quasiregular.* If α is not quasiregular choose $\delta > 0$ such that $\delta\alpha$ lies on $\bar{F}(x, \alpha) = 1$. Then m, $2 \leq m \leq n$, regular directions $\beta_1, \cdots, \beta_m$ exist such that

$$\delta\alpha = \Sigma \delta_j \beta_j, \qquad \delta_j > 0, \qquad \Sigma \delta_i = 1 \tag{12}$$

For $F(x, \alpha) = 1$ is connected, hence n points $\beta_1, \cdots, \beta_n$ on $F(x, \alpha) = 1$ exist such that the $(n-1)$-simplex spanned by $\beta_1, \cdots, \beta_n$ contains $\delta\alpha$ (see [1, p. 9]). If $\delta\alpha$ is not in the interior of the simplex choose the notation so that the $(m-1)$-simplex spanned by $\beta_1, \cdots, \beta_m$ contains $\delta\alpha$ in its interior. There is a hyperplane through $\delta\alpha$ which is simultaneously a supporting plane P of $F(x, \alpha) = 1$ and $\bar{F}(x, \alpha) = 1$ (see [1, p. 6]). Then P cannot separate any two of the points $\beta_1, \cdots, \beta_n$ and must therefore contain all the points $\beta_1, \cdots, \beta_m$, so that all these points are regular. This proves (12).

Applying this result to $\alpha_i = x(t_{i+1}) - x(t_i)$ yields relations of the form

$$\alpha_i = \sum_j \delta_{ij} \beta_j, \qquad \delta_{ij} > 0.$$

Because F is homogeneous and the simplex spanned by $\beta_{i1} \cdots, \beta_{im_i}$ lies by construction on $\bar{F}(x, \alpha) = 1$,

$$\bar{F}(x(t_i), \alpha_i) = \sum_j \bar{F}(x(t_i), \delta_{ij}, \beta_{ij}) = \sum_k \bar{F}(x(t_i), y_{ik+1} - y_{ik})$$

where y_{ik} is the point $x(t_i) + \sum_{j=1}^{k} \delta_{ij}\beta_{ij}$. Since all the β_{ik} are quasiregular

$$F(x(t_i), y_{ik+1} - y_{ik}) = \bar{F}(x(t_i), y_{ik+1} - y_i).$$

If $p(t)$ is the euclidean polygon with vertices

$$y_{00} = x(a), y_{01}, \cdots, y_{0m_0} = x(t_1), y_1, \cdots, y_{1m_1} = x(t_2), \cdots, y_{Nm_N} = x(b)$$

then $\int F(p, \dot{p})\,dt$ will, for sufficiently small norm of $[t]$, differ arbitrarily little from

$$\sum_i \sum_k F(x(t_i), y_{ik+1} - y_{ik}) = \sum \bar{F}(x(t_i), x(t_{i+1}) - x(t_i)).$$

Hence, as the norm of $[t]$ tends to 0, the curve $p(t)$ tends to $x(t)$ in the sense of Fréchet and

$$\int F(p, \dot{p})\,dt \to \int \bar{F}(x, \dot{x})\,dt$$

which proves the theorem, because $\lambda \frac{L}{F}(x) \leq \lim \int F(p, \dot{p})\,dt$.

A consequence of Theorem 2, which is contained in the results of [3] and which can be generalized to planes as minimal surfaces in Minkowski spaces is this:

For any integrand $F(\alpha)$ in a Minkowski space the straight lines are shortest connections of their endpoints for Lebesgue F-length.

This is true since they are shortest connections for $\bar{F}$-length.

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CORRECTIONS.

By I. N. Kagno.

The author wishes to make the following corrections to his paper "Linear Graphs of Degree ≤ 6 and their Groups," this Journal, vol. LXVIII, pp. 505-520.

Page 507, line 28 should read

"Suppose that $I^{a_1}_{a_1+i} = 1, \cdots$".

Page 514, line 14 should read

$H_5 \equiv [ab, ad, ae, af, bc, be, bf, cd, ce, cf, de, df]$.

Page 514, Theorem 3.4 should read

H_5 has the group $\mathfrak{H}_{32} \equiv (abcdef)_{48}$.

Page 520, Theorem D. 12 should read

$\mathfrak{H}_{32} \equiv$ *has the graph* H_5 (Theorem 3.4).

Page 520, delete lines 20 and 21.

Page 520, line 10 should read

The group $\mathfrak{H}_{19}$ has the graph H_4. (Theorem 3.3).